RADIUS OF STARLIKENESS OF CONVEX COMBINATIONS OF UNIVALENT STARLIKE FUNCTIONS

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ABSTRACT. The radius of starlikeness of the convex combination

$$tf(z) + (1-t)g(z), \quad 0 < t < 1,$$

where $f(z)$ and $g(z)$ are normalized univalent starlike functions, is $r_u = 0.4035 \dots$, the positive root of the equation $r^6 + 5r^4 + 79r^2 - 13 = 0$.

Let S denote the class of functions that are normalized and univalent in the unit disk $E = \{z: |z| < 1\}$. Let St , K be the subclasses of S of functions that map E onto starlike and convex domains respectively. Robertson [3] showed that the radius of univalence of the class

$$F = \{h(z) = tf(z) + (1-t)g(z): f(z), g(z) \in St, 0 < t < 1\}$$

is r_u , where $r_u = 0.4035 \dots$ is the positive root of the equation $r^6 + 5r^4 + 79r^2 - 13 = 0$. Labelle and Rahman [2] found that for $f, g \in St(K)$, the radius of starlikeness (convexity) of $\frac{1}{2}(f(z) + g(z))$ is greater than r_0 , where $r_0 = 0.3966 \dots$ is the smallest positive root of the equation $1 - 3r + 2r^2 - 2r^3 = 0$. Their method yields the same result when applied to the general convex combinations.

In this note, a sharp solution to Labelle and Rahman's problem is given. In fact, we make use of results of [3], [4] to prove that the radius of starlikeness of F and that of the closed convex hull of starlike functions, $\overline{\text{co}} St$, are both equal to r_u .

Let A_0 be the set of functions $f(z)$ regular in E and normalized by $f(0) = 1$. Let A_1 be the set $\{zf: f \in A_0\}$. For any set $V \subseteq A_1$, we denote by $\hat{V}$ the set $\{f/z: f \in V\}$. The dual U^* of any set $U \subseteq A_0$ is defined to be

$$U^* = \{f \in A_0: (f * g)(z) \neq 0 \text{ for all } g \in U, z \in E\},$$

where $f * g$ is the Hadamard product of f and g , i.e., if

$$f(z) = \sum_{k=0}^{\infty} a_k z^k, \quad g(z) = \sum_{k=0}^{\infty} b_k z^k$$

then

$$(f * g)(z) = \sum_{k=0}^{\infty} a_k b_k z^k.$$

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Also, by U^{**} we mean $(U^*)^*$. We then have

LEMMA (RUSCHEWEYH [4]). Let $V, W \subseteq A_0$ and $g \in A_0$. Let

$$r_1 = \sup\{r: (g * f)(rz) \in V^* \text{ for all } f \in W^{**}\},$$

$$r_2 = \sup\{r: (g * f)(rz) \in W^* \text{ for all } f \in V\}.$$

If V is compact and if $f(xz) \in V$ for all $f \in V$ and $|x| \leq 1$, then $\min(1, r_1) = \min(1, r_2)$.

THEOREM. The radius of starlikeness of $\widehat{\text{co St}}$ is r_u , where $r_u = 0.4035 \dots$ is the positive root of the equation $r^6 + 5r^4 + 79r^2 - 13 = 0$.

PROOF. Let $d(z) = \sum_{k=0}^{\infty} (k+1)z^k$, then for $f \in A_0$, $(zf)' = d * f$. Hence from Brickman, MacGregor and Wilken [1] we can write

$$\widehat{\text{co St}} = \{d * f: f \in P_{1/2,0}\},$$

where $P_{1/2,0} = \{f \in A_0: \operatorname{Re} f(z) > \frac{1}{2}, z \in E\}$. Thus the radius of starlikeness r_s of $\widehat{\text{co St}}$ is given by

$$r_s = \sup\{r: (d * f)(rz) \in \widehat{\text{St}} \text{ for all } f \in P_{1/2,0}\}.$$

Further, if we let

$$P_0 = \{f \in A_0: \exists \alpha \in \mathbf{R}, \operatorname{Re}\{e^{i\alpha}f(z)\} > 0, z \in E\},$$

$$X = \left\{ z \frac{1 + \frac{1}{2}(\xi - \eta)z}{(1 - \eta z)^2} : (|\eta| = |\xi| < 1) \text{ or } (\xi = \eta \text{ and } |\xi| < 1) \right\},$$

then, from Ruscheweyh [4], $P_0^* = P_{1/2,0}$, $P_0 = P_{1/2,0}^*$, $\hat{X}^* = \widehat{\text{St}}$ and $\hat{X}$ is compact with $f(xz) \in \hat{X}$ for all $f \in \hat{X}$, $|x| \leq 1$. Therefore, from the Lemma, we find, on putting $V = \hat{X}$, $W = P_{1/2,0}$, that

$$r_s = \sup\{r: (d * f)(rz) \in P_0 \text{ for all } f \in \hat{X}\},$$

or equivalently,

$$r_s = \sup\{r: f'(rz) \in P_0 \text{ for all } f \in X\}.$$

Since X is a subset of S ,

$$r_s \geq \sup\{r: f'(rz) \in P_0 \text{ for all } f \in S\}.$$

Now, according to Robertson [3], there exists $\alpha \in \mathbf{R}$ such that $\operatorname{Re}\{e^{i\alpha}f'(z)\} > 0$ for $|z| < r$, $0 < r < 1$, $f \in S$ if and only if, for every β , $h(z) = tf(z) + (1 - t)e^{-i\beta}f(e^{i\beta}z)$, $0 < t < 1$, is univalent in $|z| < r$; furthermore, $h(z)$ is univalent in $|z| < r_u$. Thus we have $r_s \geq r_u$. On the other hand, the radius of univalence of $\widehat{\text{co St}}$, which is also r_u , is greater than or equal to its radius of starlikeness r_s . Hence $r_s = r_u$.

The extremal function is as given by Robertson [3],

$$f_0(z) = \frac{1}{2} \frac{z}{(1 - z)^2} + \frac{1}{2} \frac{z}{(1 - ze^{-2i\alpha})^2},$$

where

$$\cos \alpha = \frac{(1 + 3r_u^4)^{1/2} + r_u^2 - 1}{2r_u}.$$

We note that since $f_0(z) \in F \subseteq \overline{\text{co St}}$, r_u is also the radius of starlikeness of F .

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