PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 78, Number 1, January 1980

OSCILLATION OF FIRST-ORDER NONLINEAR DIFFERENTIAL EQUATIONS WITH DEVIATING ARGUMENTS

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ABSTRACT. This paper is devoted to the study of the oscillatory behavior of solutions of the first-order nonlinear functional differential equations

$$x'(t) = \sum_{i=1}^{N} q_i(t) f_i(x(g_i(t))) + F(t, x(t), x(g_1(t)), \ldots, x(g_N(t))), \tag{A}$$

$$x'(t) + \sum_{i=1}^{N} q_i(t) f_i(x(g_i(t))) + F(t, x(t), x(g_1(t)), \ldots, x(g_N(t))) = 0. \tag{B}$$

First, without assuming that the deviating arguments $g_i(t)$, $1 \leqslant i \leqslant N$, are retarded or advanced, sufficient conditions are established for all solutions of (A) and (B) to be oscillatory.

Secondly, a characterization of oscillation of all solutions is obtained for equation (A) with $F \equiv 0$ and $g_i(t) > t$, $1 \leqslant i \leqslant N$, as well as for equation (B) with $F \equiv 0$ and $g_i(t) < t$, $1 \leqslant i \leqslant N$.

The purpose of this paper is to obtain oscillation criteria for the first order differential equations

$$x'(t) = \sum_{i=1}^{N} q_i(t) f_i(x(g_i(t))) + F(t, x(t), x(g_1(t)), \ldots, x(g_N(t))), \tag{A}$$

$$x'(t) + \sum_{i=1}^{N} q_i(t) f_i(x(g_i(t))) + F(t, x(t), x(g_1(t)), \ldots, x(g_N(t))) = 0, \tag{B}$$

where the following conditions are assumed to hold:

(a) $q_i, g_i \in C[[a, \infty), R]$, $q_i(t) \geqslant 0$, and $\lim_{t\to\infty} g_i(t) = \infty$, $1 \leqslant i \leqslant N$;

(b) $f_i \in C[R, R]$, f_i is nondecreasing, and $uf_i(u) > 0$ for $u \neq 0$, $1 \leqslant i \leqslant N$;

(c) $F \in C[[a, \infty) \times R^{N+1}, R]$, and $u_0 F(t, u_0, u_1, \ldots, u_N) \geqslant 0$ for $u_0 u_i > 0$, $1 \leqslant i \leqslant N$.

In what follows, by a proper solution of (A) or (B), we mean a function $x \in C^1[[T_x, \infty), R]$ which satisfies (A) or (B) for all sufficiently large t and $\sup\{|x(t)|: t \geqslant T\} > 0$ for any $T \geqslant T_x$. The standing hypothesis is that equations (A) and (B) do possess proper solutions. A proper solution of (A) or

Received by the editors January 5, 1979.
AMS (MOS) subject classifications (1970). Primary 34K15, 34C10.
Key words and phrases. First-order differential equation, nonlinear differential equation, deviating argument, oscillatory solution, oscillation.

0002-9939/80/0000-0016/$02.25

(B) is called oscillatory if it has arbitrarily large zeros and it is called nonoscillatory otherwise.

The main results of this paper are as follows.

THEOREM 1. *Suppose that each* f_i, $1 \leqslant i \leqslant N$, *satisfies*

$$\int_M^{\infty} \frac{du}{f_i(u)} < \infty \quad \textit{and} \quad \int_{-M}^{-\infty} \frac{du}{f_i(u)} < \infty \quad \textit{for any } M > 0. \tag{1}$$

All proper solutions of (A) *are oscillatory if*

$$\sum_{i=1}^{N} \int_{\mathscr{A}_i} q_i(t)\, dt = \infty, \tag{2}$$

where $\mathscr{A}_i = \{t \in [a, \infty) \colon g_i(t) > t\}$, *the advanced part of* $g_i(t)$.

THEOREM 2. *Suppose that each* f_i, $1 \leqslant i \leqslant N$, *satisfies*

$$\int_0^{m} \frac{du}{f_i(u)} < \infty \quad \textit{and} \quad \int_0^{-m} \frac{du}{f_i(u)} < \infty \quad \textit{for any } m > 0. \tag{3}$$

All proper solutions of (B) *are oscillatory if*

$$\sum_{i=1}^{N} \int_{\mathscr{R}_i} q_i(t)\, dt = \infty, \tag{4}$$

where $\mathscr{R}_i = \{t \in [a, \infty) \colon a \leqslant g_i(t) < t\}$, *the retarded part of* $g_i(t)$.

All the literature on the oscillation of first-order functional differential equations has been devoted to the case where the deviating arguments involved are retarded or advanced (see, for example, [**1**]–[**10**]), and so the above theorems can be covered by none of the previous results.

PROOF OF THEOREM 1. Let $x(t)$ be a nonoscillatory solution which is eventually positive. There is $T > a$ such that $x(t) > 0$ and $x(g_i(t)) > 0$ for $t \geqslant T$, $1 \leqslant i \leqslant N$. By conditions (b) and (c), $f_i(x(t)) > 0$, $1 \leqslant i \leqslant N$, and $F(t, x(t), \ldots) \geqslant 0$ on $[T, \infty)$, and so from (A), $x'(t) > 0$ for $t \geqslant T$, which implies that the $f_i(x(t))$ are nondecreasing on $[T, \infty)$. Let i be fixed. We divide (A) by $f_i(x(t))$ and integrate it on $[T, T']$, $T' > T$. Using condition (c) and noting that $f_i(x(g_i(t))) \geqslant f_i(x(t))$ for $t \in \mathscr{A}_i \cap [T, T']$, we then have

$$\begin{aligned} \int_T^{T'} \frac{x'(t)}{f_i(x(t))}\, dt &\geqslant \int_T^{T'} q_i(t) \frac{f_i(x(g_i(t)))}{f_i(x(t))}\, dt \\ &\geqslant \int_{\mathscr{A}_i \cap [T, T']} q_i(t)\, dt. \end{aligned} \tag{5}$$

Letting $T' \to \infty$ in (5) and taking (1) into account, we find

$$\int_{\mathscr{A}_i \cap [T, \infty)} q_i(t)\, dt \leqslant \int_{x(T)}^{x(\infty)} \frac{du}{f_i(u)} < \infty.$$

Since i is arbitrary, this contradicts (2), and hence (A) cannot have eventually positive proper solutions. Similarly, (A) does not possess eventually negative proper solutions.

Proof of Theorem 2. Let $x(t)$ be a nonoscillatory solution of (B). Without loss of generality we may suppose that $x(t)$ is eventually positive. There is $t_0 > a$ such that $x(t) > 0$ and $x(g_i(t)) > 0$ for $t \geqslant t_0$, $1 \leqslant i \leqslant N$. Take $T > t_0$ so large that $g_i(t) \geqslant t_0$ for $t \geqslant T$, $1 \leqslant i \leqslant N$. Since $x'(t) < 0$, $t \geqslant t_0$, by (B), the $f_i(x(t))$ are positive and nonincreasing on $[t_0, \infty)$, so that $f_i(x(g_i(t))) \geqslant f_i(x(t))$ for $t \in \mathfrak{R}_i \cap [T, T']$. Proceeding as in the proof of Theorem 1, we obtain from (B)

$$\int_T^{T'} \frac{-x'(t)}{f_i(x(t))}\,dt \geqslant \int_T^{T'} q_i(t)\frac{f_i(x(g_i(t)))}{f_i(x(t))}\,dt$$
$$\geqslant \int_{\mathfrak{R}_i \cap [T,T']} q_i(t)\,dt. \tag{6}$$

Letting $T' \to \infty$ in (6) and using (3), we see that

$$\int_{\mathfrak{R}_i \cap [T,\infty)} q_i(t)\,dt \leqslant \int_{x(\infty)}^{x(T)} \frac{du}{f_i(u)} < \infty$$

for $1 \leqslant i \leqslant N$, which contradicts (4). This completes the proof.

Remark. If $g_i(t) > t$, $1 \leqslant i \leqslant N$ (resp. $g_i(t) < t$, $1 \leqslant i \leqslant N$), then condition (2) (resp. (4)) reduces to

$$\sum_{i=1}^{N} \int^{\infty} q_i(t)\,dt = \infty. \tag{7}$$

Thus Theorem 1 is an extension of a result of Anderson [**1**, Theorem 3].

We now consider the particular cases of (A) and (B).

$$x'(t) = \sum_{i=1}^{N} q_i(t) f_i(x(g_i(t))), \tag{A$_0$}$$

$$x'(t) + \sum_{i=1}^{N} q_i(t) f_i(x(g_i(t))) = 0. \tag{B$_0$}$$

A sufficient condition for (A_0) and (B_0) to have nonoscillatory solutions is given in the following theorem.

Theorem 3. *Let conditions* (a) *and* (b) *hold. If*

$$\sum_{i=1}^{N} \int^{\infty} q_i(t)\,dt < \infty, \tag{8}$$

then equations (A_0) *and* (B_0) *have nonoscillatory solutions.*

Proof. For an arbitrarily given constant $k > 0$, consider the integral equation

$$x(t) = k + \sum_{i=1}^{N} \int_T^t q_i(s) f_i(x(g_i(s)))\,ds, \tag{9}$$

where $T > a$ is chosen so that

$$\sum_{i=1}^{N} f_i(2k) \int_T^{\infty} q_i(s)\,ds < k.$$

Put $T_0 = \min_{1 \leqslant i \leqslant N} \inf_{t \geqslant T} g_i(t)$ and let C denote the locally convex space of all continuous functions $x\colon [T_0, \infty) \to R$ with the topology of uniform convergence on compact subintervals of $[T_0, \infty)$. Let $X = \{x \in C\colon k \leqslant x(t) \leqslant 2k,\ t \geqslant T_0\}$. Define the operator $\Phi\colon X \to C$ by

$$\Phi x(t) = k + \sum_{i=1}^{N} \int_T^t q_i(s) f_i(x(g_i(s)))\, ds, \quad t \geqslant T,$$

$$\Phi x(t) = k, \quad T_0 \leqslant t \leqslant T. \tag{10}$$

It is easy to verify that Φ maps X, which is a closed convex subset of C, continuously into a compact subset of X. Consequently, by the Tychonoff fixed-point theorem, Φ has a fixed point x in X. Obviously, this fixed point $x = x(t)$ satisfies (9) for $t \geqslant T$ and hence becomes a nonoscillatory solution of (A_0).

Similarly, a nonoscillatory solution of (B_0) is obtained as a solution to the integral equation

$$x(t) = 2k - \sum_{i=1}^{N} \int_T^t q_i(s) f_i(x(g_i(s)))\, ds.$$

It would be of interest to observe that by combining Theorems 1 and 2 with Theorem 3 one easily obtains a characterization of oscillation of (A_0) in the advanced case and equation (B_0) in the retarded case.

THEOREM 4. *Suppose that* (1) *holds and that* $g_i(t) > t$, $1 \leqslant i \leqslant N$. *Then* (7) *is a necessary and sufficient condition for all proper solutions of* (A_0) *to be oscillatory.*

THEOREM 5. *Suppose that* (3) *holds and that* $g_i(t) < t$, $1 \leqslant i \leqslant N$. *Then* (7) *is a necessary and sufficient condition for all proper solutions of* (B_0) *to be oscillatory.*

REMARK. Theorem 5 was first proved by Koplatadze **[2]**.

EXAMPLE. Consider the equation

$$x'(t) = \frac{|x(t + \sin t)|^{\alpha} \operatorname{sgn} x(t + \sin t)}{t^{\beta}[\log(t + \sin t)]^{\alpha}}, \quad t \geqslant 2\pi, \tag{11}$$

where $\alpha > 0$ and β are real constants. The advanced part of $g(t) = t + \sin t$ is $\mathcal{Q} = \bigcup_{k=1}^{\infty} (2k\pi, (2k+1)\pi)$.

(i) Let $\alpha > 1$. If $\beta < 1$, then

$$\int_{\mathcal{Q}} \frac{dt}{t^{\beta}[\log(t + \sin t)]^{\alpha}} = \sum_{k=1}^{\infty} \int_{2k\pi}^{(2k+1)\pi} \frac{dt}{t^{\beta}[\log(t + \sin t)]^{\alpha}} = \infty, \tag{12}$$

and so from Theorem 1 it follows that all proper solutions of (11) are oscillatory. If $\beta \geqslant 1$, then

$$\int_{2\pi}^{\infty} \frac{dt}{t^{\beta}[\log(t + \sin t)]^{\alpha}} < \infty,$$

and hence, by Theorem 3, (11) has bounded nonoscillatory solutions. In this case (11) may have unbounded nonoscillatory solutions; in fact, $x(t) = \log t$ is such a solution when $\beta = 1$.

(ii) Let $0 < \alpha \leqslant 1$ and $\beta = 1$. Then (12) holds, but (11) has a nonoscillatory solution $x(t) = \log t$. This example shows that the conclusion of Theorem 1 is not true if condition (1) is violated.

A similar example illustrating Theorem 2 could easily be provided.

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