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ERGODIC PROJECTIONS OF CONTINUOUS AND DISCRETE SEMIGROUPS

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ABSTRACT. Let X be a Banach space. Let $\{T(t);\ t > 0\}$ be a uniformly bounded semigroup of operators on X, which converges strongly to P, known to be a projection, as t goes to 0. If A is its generator and X_0 [resp., X_t, $t > 0$] is the set of x for which

$$P_0x \equiv \lim_{t\to\infty} t^{-1}\int_0^t T(\tau)x\,d\tau \quad \left[\text{resp., } P_tx \equiv \lim_{n\to\infty} n^{-1}\sum_{i=0}^{n-1} T(it)x\right]$$

exists, then, for each $t > 0$, P_t is a bounded projection in X_t; when $t = 0$, $X_0 = N(A) \oplus \overline{R(A)} \oplus N(P)$, $R(P_0) = N(A)$ and $N(P_0) = \overline{R(A)} \oplus N(P)$; when $t > 0$, then

$$X_t = N(T(t) - I) \oplus \overline{R(T(t) - I)},$$

$R(P_t) = N(T(t) - I)$ and $N(P_t) = \overline{R(T(t) - I)}$; $X_t = X$ for all $t > 0$ if X is reflexive. Some results on relations among the projections P_t, $t > 0$, are obtained. In particular, we have $P_t = P_0$ for all sufficiently small t if A is bounded.

1. Introduction. Let $S = \{T(t);\ t > 0\}$ be a semigroup of bounded linear operators on a Banach space X. The infinitesimal generator A of S is defined as $Ax \equiv \lim_{t\to 0}[T(t) - I]x/t$ wherever the limit exists. Let $N(A)$ and $R(A)$ denote the null space and range, respectively, of A. Also we denote by $R(S)$ the set $\{T(t)x;\ x \in X, t > 0\}$.

If S satisfies the additional assumption that

$$Px = \lim_{t\to 0} T(t)x \quad \text{exists for all } x \in X, \tag{1}$$

then P is a projection from X onto $\overline{R(S)}$, and $T(t) = PT(t) = T(t)P$ (see [3, p. 319]). Thus, we have $T(t)|\overline{R(S)} = T(t)P$, $T(t)|N(P) = 0$ and

$$T(t) = \begin{bmatrix} T(t)|\overline{R(S)} & 0 \\ 0 & 0 \end{bmatrix}.$$

$T(t)$ is strongly continuous; in fact, $T(t)|\overline{R(S)}$ is a semigroup of class (C_0) in $\overline{R(S)}$ with generator $A|\overline{R(S)} = A$. It follows that A is a closed operator with $\overline{D(A)} = \overline{R(S)}$ and $R(A) \subset \overline{R(S)}$. The following extension of Theorem 1 with Corollary 2 of [4] (or cf. [2]) is obtained immediately.

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THEOREM 1.1. *If S is bounded by $M \geqslant 1$ and if it satisfies* (1), *then the set X_0 of all $x \in X$ such that*

$$P_0 x \equiv \lim_{t\to\infty} t^{-1}\int_0^t T(\tau)x\,d\tau = \lim_{\lambda\to 0^+} \lambda(\lambda I - A)^{-1}Px \tag{2}$$

exists is a closed subspace; $X_0 = N(A) \oplus \overline{R(A)} \oplus N(P)$. *$P_0$ is a projection in X_0, with range $N(A)$ and null space $\overline{R(A)} \oplus N(P)$, and has norm $\|P_0\| \leqslant M$. In general, $N(A) \oplus \overline{R(A)} \subset \overline{R(S)}$; but, if X is reflexive, then they are identical and therefore $X_0 = X$.*

Using the notation $A_t \equiv [T(t) - I]/t$, we have $\|\exp(\tau A_t)\| \leqslant M$ for all $t > 0$ and for all $\tau \geqslant 0$ if $\|T(t)\| \leqslant M$ for all $t > 0$. (See Corollary 2.2.) Now, by Theorem 1.1, we can associate with each $t > 0$ a closed subspace $X_t \equiv N(A_t) \oplus \overline{R(A_t)}$ and a projection P_t: $X_t \to X_t$ which is defined as

$$P_t x \equiv \lim_{u\to\infty} \frac{1}{u}\int_0^u \exp(\tau A_t)x\,d\tau, \qquad x \in X_t.$$

We have $R(P_t) = N(A_t)$, $N(P_t) = \overline{R(A_t)}$ and $\|P_t\| \leqslant M$.

An interesting phenomenon is that for each $t > 0$, P_t, defined as above, is precisely the same as that defined by

$$P_t x = \lim_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} T(it)x. \tag{3}$$

This results from Theorem 1 of **[6**, p. 213] on the limit problem

$$Jx \equiv \lim_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} T^i x \tag{4}$$

for a general uniformly bounded discrete semigroup $\{T^n;\ n = 0, 1, \ldots\}$. Actually, this can also be derived from Theorem 1.1, which will be done in §2. §3 is concerned with relations among the projections P_t, $t \geqslant 0$.

2. Ergodic theorems for discrete semigroups. Let $T \in B(X)$. Its spectral radius $r_\sigma(T) \equiv \max\{|\lambda|;\ \lambda \in \sigma(T)\}$ is equal to $\lim_{n\to\infty}(\|T^n\|)^{1/n} \leqslant \|T\|$. Hence, for any $r > r_\sigma(T)$, there is an $M \geqslant 1$ such that $\|T^n\| \leqslant Mr^n$ for all $n = 0, 1, 2, \ldots$.

Conversely, we have

LEMMA 2.1. *If $\|T^n\| \leqslant Mr_0^n$, $n = 0, 1, 2, \ldots,$ then every λ with $|\lambda| > r_0$ belongs to the resolvent set $\rho(T)$, i.e., $r_\sigma(T) \leqslant r_0$, and the following inequality holds.*

$$\|\exp(tT)\| \leqslant M\exp(tr_0), \qquad t > 0. \tag{5}$$

The proof is just routine arguments on the convergence of the Neumann series and the norm estimate of the series expansion of $\exp(tT)$. The details will be omitted here.

If we replace t, T and r_0 by t/τ, $T(\tau)$ and $e^{w\tau}$ respectively, then we have

COROLLARY 2.2. *If* $\|T(t)\| \leqslant Me^{wt}$ *for all* $t \geqslant 0$, *then*

$$\|\exp(tA_\tau)\| \leqslant M \exp\left(t\frac{e^{w\tau}-1}{\tau}\right)$$

for all $\tau > 0$ *and for all* $t \geqslant 0$.

Our key result in this section is

THEOREM 2.3. *If* $\|T^n\| \leqslant M$ *for some* $M \geqslant 1$ *and all* $n = 0, 1, 2, \ldots,$ *then the limit* $\lim_{n\to\infty}(1/n)\sum_{i=0}^{n-1} T^i x$ *exists if and only if the limit* $\lim_{\lambda\to 0^+} \lambda[\lambda I - (T-I)]^{-1}x$ *exists, and they are identical when they exist.*

The proof uses the following:

PROPOSITION. *Let* $f(t)$ *be a bounded and strongly measurable* X*-valued function defined on* $(0, \infty)$. *Then*

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t f(\tau)\, d\tau = \lim_{\mu\to 0^+} \mu \int_0^\infty e^{-\mu t} f(t)\, dt,$$

provided either limit exists.

For the proof, see [5, Theorems 8.2.3 and 8.2.4].

PROOF OF THEOREM 2.3. We put $f(t) = T^i x$ for $i < t \leqslant i+1$, $i = 0, 1, \ldots.$ Then the following computation verifies the assertion.

$$\begin{aligned}
\lim_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} T^i x &= \lim_{n\to\infty} \frac{1}{n}\int_0^n f(t)\, dt = \lim_{\mu\to 0^+} \mu \int_0^\infty e^{-\mu t} f(t)\, dt \\
&= \lim_{\mu\to 0^+} \mu \sum_{i=0}^{\infty} \int_i^{i+1} e^{-\mu t} T^i x\, dt \\
&= \lim_{\mu\to 0^+} \sum_{i=0}^{\infty} \left[e^{-i\mu} - e^{-(i+1)\mu}\right] T^i x \\
&= \lim_{\mu\to 0^+} (e^\mu - 1) \sum_{i=0}^{\infty} e^{-(i+1)\mu} T^i x \\
&= \lim_{\mu\to 0^+} (e^\mu - 1)(e^\mu I - T)^{-1} x \\
&= \lim_{\lambda\to 0^+} \lambda\left[\lambda I - (T-I)\right]^{-1} x.
\end{aligned}$$

The invertibility of $\lambda I - (T-I)$ follows from Lemma 2.1.

Since $\|T^n\| \leqslant M$ implies $\|\exp(t(T-I))\| \leqslant M$ for all $t \geqslant 0$, Theorem 1.1 applies with $A = T - I$ and $P = I$ so that the limits in Theorem 2.3 are also equal to the limit $\lim_{t\to\infty}(1/t)\int_0^t e^{\tau(T-I)}x\, d\tau$. Therefore we have the following mean ergodic theorem for discrete semigroups.

THEOREM 2.4. *If* $\|T^n\| \leqslant M$ *for all* $n = 0, 1, 2, \ldots,$ *then the set* X_J *of all* $x \in X$ *such that*

$$Jx \equiv \lim_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} T^i x = \lim_{\lambda\to 0^+} \lambda[\lambda I - (T-I)]^{-1}x$$
$$= \lim_{t\to\infty}\frac{1}{t}\int_0^t e^{\tau(T-I)}x\,d\tau$$

exists is $N(T-I) \oplus \overline{R(T-I)}$. J *is a projection in* X_J *with range* $N(T-I)$, *null space* $\overline{R(T-I)}$, *and with norm* $\|J\| \leqslant M$. *When* X *is reflexive, we have* $X_J = X$.

3. The projections P_t. Throughout this section, let $S = \{T(t); t > 0\}$ satisfy the assumptions in Theorem 1.1, and let $\{P_t;\ t \geqslant 0\}$ be the projections defined by (2) and (3). That is, if $t = 0$, then $X_0 \equiv N(A) \oplus \overline{R(A)} \oplus N(P)$, $R(P_0) = N(A)$ and $N(P_0) = \overline{R(A)} \oplus N(P)$; if $t > 0$, then $X_t \equiv N(A_t) \oplus \overline{R(A_t)}$, $R(P_t) = N(A_t)$ and $N(P_t) = \overline{R(A_t)}$. In this section we will study relations among P_t.

We begin with three lemmas which will be used in the proofs of the subsequent theorems.

LEMMA 3.1. x *belongs to* X_0 *if and only if* $\lim_{n\to\infty}(1/t_n)\int_0^{t_n} T(\tau)x\,d\tau$ *exists for some increasing sequence of positive numbers* $\{t_n\}$ *which diverges to infinity and satisfies the condition that* $a = \sup_n(t_{n+1} - t_n) < \infty$. *For any such* x *and* $\{t_n\}$, *the limit is* $P_0 x$.

In particular, the assertion holds when $t_n = nt$, which is the case we will need in the proof of Theorem 3.4.

PROOF. The necessity is trivial. To show the sufficiency, let $\{t_n\}$ satisfy the required condition and y be the limit. Since for any t, there is an integer n such that $t_{n-1} \leqslant t \leqslant t_n$, therefore, we have the estimate

$$\left\|\frac{1}{t}\int_0^t T(s)x\,ds - \frac{1}{t_n}\int_0^{t_n} T(s)x\,ds\right\|$$
$$\leqslant \left|\frac{t_n}{t} - 1\right| \left\|\frac{1}{t_n}\int_0^{t_n} T(s)x\,ds\right\| + \left\|\frac{1}{t}\int_t^{t_n} T(s)x\,ds\right\|$$
$$\leqslant 2\frac{(t_n - t)}{t} M\|x\| \leqslant \frac{2aM\|x\|}{t}$$

which goes to 0 as $t \to \infty$. This means that $x \in X_0$ and $y = P_0 x$.

LEMMA 3.2. *The following statements are equivalent.*

(a) $x \in N(A)$.

(b) $T(t)x = x$ *for all* $t > 0$.

(c) $T(t)x = x$ *for all* t *in some infinite set* U *of positive numbers which has at least one limit point*.

(d) *There is a positive sequence* $\{t_n\}$ *such that* $t_n \to 0$ *and such that* $T(t_n)x = x$ *for all* n.

PROOF. (a) $\Rightarrow$ (b). If $x \in N(A) \subset \overline{R(S)} = R(P)$, then $dT(t)x/dt = T(t)Ax = 0$ for all $t > 0$. So,

$$T(t)x - x = T(t)x - Px = \lim_{\varepsilon \to 0^+} \left[T(t)x - T(\varepsilon)x \right]$$

$$= \lim_{\varepsilon \to 0^+} \int_\varepsilon^t \frac{d}{d\tau} T(\tau)x \, d\tau = 0$$

for all $t > 0$. "(b) $\Rightarrow$ (a)" follows from the definition of A, and "(b) $\Rightarrow$ (c)" is trivial. To prove (c) $\Rightarrow$ (d), let $\{s_n\} \subset U$ and $s_n \to s_0 > 0$. Then, $T(s_0)x = \lim_{n \to \infty} T(s_n)x = x$ by the strong continuity of $T(t)$. On taking $t_n = |s_n - s_0|$, we have $t_n \to 0$ and $T(t_n)x = x$, $n = 1, 2, \ldots$. In fact, if $s_n > s_0$, then

$$T(t_n)x = T(t_n)T(s_0)x = T(s_n)x = x;$$

if $s_n < s_0$, then $T(t_n)x = T(t_n)T(s_n)x = T(t_n + s_n)x = T(s_0)x = x$. Next, suppose (d) holds. Since the set of positive t such that $T(t)x = x$ is a relatively closed sub-semigroup of $(0, \infty)$, and since it contains arbitrarily small numbers, it is all of $(0, \infty)$, i.e., (b) is true.

LEMMA 3.3. *If C_t denotes the operator $t^{-1} \int_0^t T(\tau)\, d\tau$, then*

$$C_{nt}x = C_t \frac{1}{n} \sum_{i=0}^{n-1} T(it)x = \left[\frac{1}{n} \sum_{i=0}^{n-1} T(it) \right] C_t x \tag{6}$$

for all $x \in X$, $t > 0$ and $n = 1, 2, \ldots$.

PROOF.

$$C_{nt}x = \frac{1}{n} \int_0^n T(st)x \, ds = \frac{1}{n} \sum_{i=0}^{n-1} \int_i^{i+1} T(st)x \, ds$$

$$= \frac{1}{n} \sum_{i=0}^{n-1} T(it) \int_0^1 T(st)x \, ds$$

$$= \left[\frac{1}{n} \sum_{i=0}^{n-1} T(it) \right] C_t x,$$

and similarly for the other identity.

The following theorems and corollaries form the main results of this section.

THEOREM 3.4. *The following relations hold for all $t > 0$.*
(i) $X_t \subset X_0$, $N(P) \subset N(P_t) \subset N(P_0)$ and $R(P_t) \supset R(P_0)$.
(ii) $C_t X_0 \subset X_t$, $C_t N(P_0) \subset N(P_t)$, $C_t | R(P_t) = P_0 | R(P_t)$ and $C_t | R(P_0) = I$.
(iii) $P_0 X_t \subset X_t$, $P_0 = P_t C_t | X_0$ and $P_0 | X_t = P_t C_t | X_t = C_t P_t$.

PROOF. All the assertions follow from the two identities:

$$P_0 x = P_t C_t x \quad \text{for all } x \in X_0,$$

$$P_0 x = P_t C_t x = C_t P_t x \quad \text{for all } x \in X_t.$$

But these are obtained by taking limits of terms in (6).

COROLLARY 3.5. *$R(P_t) = R(P_0) \oplus [R(P_t) \cap N(P_0)]$. If $N(P_t) = N(P_0)$, then $R(P_t) = R(P_0)$, and so, $X_t = X_0$ and $P_t = P_0$.*

From the fact that $T(t)x = x$ for all $x \in N(A)$, and $T(t)Ax = AT(t)x$ for all $x \in D(A)$, we see that $N(A)$, $\overline{R(A)}$ and $X_0 = N(A) \oplus \overline{R(A)}$ are invariant under $T(t)$ and so under C_t, and that $C_t|\overline{R(A)}$ is a surjection if and only if $C_t|X_0$ is. Now, if they are surjective, then

$$\overline{R(A)} = C_t\overline{R(A)} = C_t\left[\,\overline{R(A)} \oplus N(P)\right] = C_tN(P_0) \subset N(P_t).$$

This, with $N(P) \subset N(P_t)$, yields the relation $N(P_0) = \overline{R(A)} \oplus N(P) \subset N(P_t) \subset N(P_0)$. Therefore, from Corollary 3.5 follows the next

COROLLARY 3.6 *If $C_t|\overline{R(A)}$ [or $C_t|X_0$] is surjective, then $X_t = X_0$ and $P_t = P_0$.*

COROLLARY 3.7. *If $\|T(t) - P\| \to 0$ as $t \to 0$ (which is equivalent to saying that the generator A is bounded), then there is a $\delta > 0$ such that $X_t = X_0$ and $P_t = P_0$ for all $0 \leqslant t < \delta$.*

PROOF. We have $\|T(t)|\overline{R(A)} - I\| \to 0$ as $t \to 0$. Let $\delta > 0$ be so chosen that $0 < t < \delta$ implies $\|T(t)|\overline{R(A)} - I\| < 1$. Then for all such t,

$$\|C_t|\,\overline{R(A)} - I\| = \left\|t^{-1}\int_0^t\left[\,T(\tau)|\,\overline{R(A)} - I\,\right]d\tau\right\| < 1.$$

Thus, $C_t|\overline{R(A)}$ is invertible, and therefore surjective. The conclusion follows immediately from Corollary 3.6.

REMARKS AND EXAMPLES. (a) If X is reflexive, then $N(P_t) = N(P_0)$ if and only if $R(P_t) = R(P_0)$.

(b) Here is an example for Corollary 3.7. Let $T(t)$ be the multiplication by e^{itx} on $C[0, 1]$. Computations show

$$\begin{aligned}(C_tf)(x) &\equiv \frac{1}{t}\int_0^t e^{i\tau x}f(x)\\ &= \begin{cases} f(0) & \text{if } x = 0,\\ f(x)(e^{itx} - 1)/itx & \text{if } x \neq 0;\end{cases}\end{aligned} \tag{7}$$

$$\begin{aligned}f_n(x) &\equiv \frac{1}{n}\sum_{k=0}^{n-1} e^{iktx}f(x)\\ &= \begin{cases} f(x) & \text{if } tx \equiv 0 \pmod{2\pi},\\ f(x)\exp\left(i\tfrac{1}{2}(n-1)tx\right)\left(\sin\tfrac{1}{2}ntx\right)/n\sin\tfrac{1}{2}(tx) & \text{elsewhere.}\end{cases}\end{aligned} \tag{8}$$

(7) tells us that if C_tf converges in $\|\cdot\|_\infty$ while $t \to \infty$, then $f(0)$ has to be 0 and the limit is the zero function. Hence we have $R(P_0) = \{0\}$ and $X_0 = N(P_0) \subset \{f \in C[0, 1];\ f(0) = 0\}$. This inclusion actually is an equality. In fact, let f be continuous and $f(0) = 0$. Then, for given $\varepsilon > 0$, there is a $\delta > 0$ such that $|f(x)| < \varepsilon$ on $[0, \delta)$, which implies $|(C_tf)(x)| < \varepsilon$ for $x \in [0, \delta)$. If

$t > 2\|f\|_\infty/\delta\varepsilon$, then

$$|(C_t f)(x)| \leqslant |(e^{itx} - 1)/itx|\ |f(x)| \leqslant 2\|f\|_\infty/t\delta < \varepsilon$$

for x in $[\delta, 1]$ also. Thus $\|C_t f\| \to 0$ as $\to \infty$, i.e., $f \in N(P_0)$.

Similarly, one can show without difficulty that $\{f_n\}$ converges uniformly if and only if $f(x) = 0$ for all those x such that $tx \equiv 0 \pmod{2\pi}$, and that the limit is 0 if it exists. That is, $R(P_t) = \{0\}$ and $X_t = N(P_t) = \{f \in C[0, 1]; f(x) = 0 \text{ if } tx \equiv 0 \pmod{2\pi}\}$. We have $X_t = X_0$ and $P_t = P_0$ for $0 \leqslant t < 2\pi$, but $X_{2\pi} = N(P_{2\pi}) = \{f \in C[0, 1]; f(0) = f(1) = 0\} \subsetneqq X_0$.

(c) The condition $\|T(t) - P\| \to 0$ is not a necessity in Corollary 3.7. For instance, the semigroup of left translations $T(t)$: $f(x) \to f(x + t)$ on $X = L^p(-\infty, \infty)$, $1 < p < \infty$, is not continuous in operator norm, but we have $X_t = X_0 = X$ and $P_t = P_0 = 0$ for all $t > 0$. First, $R(P_0) = N(A) = \{0\}$ since the spectrum of $A = d/dx$ is purely continuous [**1**, p. 37]; secondly, $R(P_t) = N(T(t) - I) = \{0\}$ since the only periodic (a.e.) function in L^p $(1 < p < \infty)$ is 0; finally, remark (a) applies.

(d) There are semigroups (with unbounded generators) such that $P_t \neq P_0$ for every $t > 0$. For instance, for the semigroup of multiplications by e^{itx} on UCB$(-\infty, \infty)$, the set of bounded uniformly continuous functions on $(-\infty, \infty)$, we have $R(P_t) = R(P_0) = \{0\}$, $X_0 = N(P_0) = \{f \in \text{UCB}(-\infty, \infty); f(0) = 0\}$ and $X_t = N(P_t) = \{f \in \text{UCB}(-\infty, \infty); f(2k\pi/t) = 0, k = 0, \pm 1, \pm 2, \dots\}$ for $t > 0$. Another example is the left translations on UCB$(-\infty, \infty)$, for which we have $R(P_0) = N(A) = \{\text{constant functions}\}$ and $R(P_t) = N(T(t) - I) = \{\text{continuous periodic functions with period } t\}$.

For general strongly continuous semigroups, we have

THEOREM 3.8. *If* $\{t_n\} \to 0$, *then* $\bigcap_{n=1}^\infty R(P_{t_n}) = R(P_0)$ *and* $\overline{\text{span}}\{N(P_{t_n}); n = 1, 2, \dots\} = N(P_0)$.

This follows from the next theorem plus the fact that $N(P_t) \subset N(P_0)$ for all $t > 0$.

THEOREM 3.9. *Let U be an infinite set of nonnegative numbers with at least one limit point. Then,*

(i) $\bigcap\{R(P_t); t \in U\} = R(P_0)$,

(ii) $[\bigcup\{N(P_t); t \in U\}]^- = [\bigcup\{N(P_t); t \in \overline{U}\}]^-$.

PROOF. $x \in R(P_t) = N(T(t) - I)$ for all $t \in U$ implies $x \in N(A) = R(P_0)$, by Lemma 3.2, therefore $\bigcap\{R(P_t); t \in U\} \subset R(P_0)$. But we also have $R(P_0) \subset R(P_t)$ for all t. So, (i) is true.

To show (ii), it suffices to show that the left side of it contains $N(P_{t_0})$ for any limit point t_0 of U. Suppose $\{t_n\} \subset U$ and $t_n \to t_0$. If $t_0 > 0$, then, for any $x \in N(P_{t_0}) = R(T(t_0) - I)^-$, there are $\{y_n\} \subset X$ such that

$$\begin{aligned} x &= \lim_{m\to\infty} (T(t_0) - I)y_m \\ &= \lim_{m\to\infty} \lim_{n\to\infty} (T(t_n) - I)y_m \in \Big[\bigcup \{N(P_{t_n}); n = 1, 2, \dots\}\Big]^- . \end{aligned}$$

If $t_0 = 0$, then for any $x \in N(P_0) = \overline{R(A)} \oplus N(P)$, there are $u \in N(P)$ and $\{w_n\} \subset D(A)$ such that

$$x = u + \lim_{m\to\infty} Aw_m = u + \lim_{m\to\infty} \lim_{n\to\infty} (T(t_n) - I)w_m/t_n$$

$$= \lim_{m\to\infty} \lim_{n\to\infty} (T(t_n) - I)(w_m/t_n - u) \in \Big[\bigcap \{N(P_{t_n}); n = 1, 2, \dots\} \Big]^- .$$

Hence, we always have $N(P_{t_0}) \subset [\bigcup \{N(P_t); t \in U\}]^-$ once t_0 belongs to $\overline{U}$. This ends the proof.

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