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A NOTE ON M-IDEALS IN $B(X)$

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ABSTRACT. In this paper we prove some properties of M-ideals and HB-subspaces in an arbitrary Banach space. We then apply these properties to prove a theorem which generalizes to other spaces Smith's and Ward's results in [8]: for $1 < p < \infty$, $B(l_p)$ contains no nontrivial summands and that each nontrivial M-ideal in $B(l_p)$ contains $K(l_p)$.

Introduction. A closed subspace J of a Banach space Y is said to be an M-ideal of Y if its annihilator $J^\perp$ is l_1 complemented in Y^*. That is, there exists a subspace J_* of Y^* such that $Y^* = J^\perp \oplus J_*$ and $\|p + q\| = \|p\| + \|q\|$ whenever $p \in J^\perp$ and $q \in J_*$. J is said to be an M-summand if J is complemented by a closed subspace J' such that $\|p + q\| = \max(\|p\|, \|q\|)$ whenever $p \in J$ and $q \in J'$. M-summands are M-ideals, though the reverse is not necessarily true. These concepts, first introduced for real Banach spaces in [1], also apply to complex Banach spaces. Recently, much interest has focused on the approximation properties of M-ideals [5], [7].

For a Banach space X, let $K(X)$ and $B(X)$ denote the spaces of compact operators and all bounded operators respectively. In [3], Hennefeld showed that for $X = c_0$ or l_p, $1 < p < \infty$, $K(X)$ is an M-ideal in $B(X)$. In [8], Smith and Ward proved that, for $1 < p < \infty$, $B(X)$ contains no nontrivial M-summands, and that any nontrivial M-ideal must contain $K(l_p)$. Their proof used Tam's characterization of Hermitian operators in $B(l_p)$, $p \neq 2$, the fact that $K(l_p)$ is the only two-sided ideal in $B(l_p)$, and their technique of investigating Banach algebra (with identity) M-ideals by looking at the associated Hermitian projections (this technique involves consideration of $B(l_p)^{**}$ and the Arens multiplication). In our proof of the generalization of the Smith-Ward result, we use instead some elementary properties of M-ideals and HB-subspaces, given in §1, and certain manipulations on matrices.

1. Some properties of M-ideals and HB-subspaces. The notion of HB-subspaces, first defined in [4], is a generalization of M-ideals. Moreover, in [4], it was shown that for certain Banach spaces $K(X)$ is only an HB-subspace, not an M-ideal, in $B(X)$.

DEFINITION 1.1. A closed subspace H of a Banach space Y is called an HB-subspace if its annihilator $H^\perp$ is complemented by a subspace H_* such that for each $f \in Y^*$, $\|f\| \geqslant \|f_\perp\|$ and $\|f\| > \|f_*\|$ whenever $f = f_* + f_\perp$ with $f_* \in H_*$ and $f_\perp$ nonzero $\in H^\perp$.

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We then have the following straightforward lemmas, some of which we merely state without proof.

LEMMA 1.2. *If H is an HB-subspace of Y, then each $\phi \in H^*$ has a unique norm-preserving extension to Y.*

LEMMA 1.3. *Let H be an HB-subspace. Then $f \in H_* \Leftrightarrow \|f/H\| = \|f\|$.*

PROOF. ($\Leftarrow$) Let f satisfy $\|f/H\| = \|f\|$. Write $f = f_* + f_\perp$. For $\varepsilon > 0$, $\exists$ norm one $x \in H$: $\|f\| - \varepsilon < f(x) = f_*(x) + f_\perp(x) = f_*(x)$. Hence, $\|f_*\| = \|f\|$, $\|f_\perp\| = 0$ and $f = f_*$.

($\Rightarrow$) For $f \in H_*$, let $g = f/H$ and $\hat{g}$ be a Hahn-Banach extension of g to Y. By the previous part of the proof, $\hat{g} \in H_*$. But $f - g$ is in both H_* and $H^\perp$, which implies $f - \hat{g} = 0$. Thus, $\|f/H\| = \|f\|$.

The proof of the above lemma shows how to obtain the decomposition for an arbitrary $g \in Y^*$, namely: g_* is the unique Hahn-Banach extension of g restricted to H, and $g_\perp = g - g_*$. Hence, we have the following lemma.

LEMMA 1.4. *If H is an HB-subspace, then H_* is isometric to H^*.*

LEMMA 1.5. *If H is an HB-subspace, and J is an M-ideal with $H_* \subset J_*$, then $H \subset J$.*

PROOF. First, we claim that $J^\perp \subset H^\perp$. To see this, suppose $g \neq 0$ is in $J^\perp$. Write $g = g_{H_*} + g_{H_\perp}$. Note that $g_{H_\perp}$ cannot be 0, since $H_* \subset J_*$; also if $g_{H_*} = 0$, then we are finished. Hence, we can suppose g_{H_*} and $g_{H_\perp}$ are both nonzero. Then,

$$\left\|-g_{H_*} + g\right\| = \|g_{H_\perp}\| < \left\|-g_{H_*}\right\| + \|g_{H_\perp}\| \quad \left(\text{since } g_{H_*} \text{ is nonzero}\right)$$

$$\leqslant \left\|-g_{H_*}\right\| + \|g\| \quad (\text{since } H \text{ is an HB-subspace}).$$

But $-g_{H_*} \in J_*$, $g \in J^\perp$ contradicts the fact that J is an M-ideal. Hence, $J^{\perp\perp} \subset H^{\perp\perp}$. Finally, $H \subset J$, since $H = H^{\perp\perp} \cap Y$, $J = J^{\perp\perp} \cap Y$.

LEMMA 1.6. *Let J be an M-ideal of Y and $f \in Y^*$. Then f is an extreme point of the unit ball of $Y^* \Leftrightarrow f$ is in J_* or $J^\perp$ and is an extreme point of the unit ball of J_* or $J^\perp$.*

LEMMA 1.7. *Let H be an HB-subspace and J an M-ideal. If $f \in H_*$ is an extreme point of the unit ball of H_*, then f is in J_* or $J^\perp$.*

2. The generalization.

DEFINITION 2.1. A basis $\{e_i\}$ is called shrinking if the biorthogonal functionals $\{e_i^*\}$ form a basis for X^*.

DEFINITION 2.2. A basis $\{e_i\}$ for a Banach space is called unconditionally monotone if $\|\Sigma_{i \in A \cup B} a_i e_i\| \geqslant \|\Sigma_{i \in A} a_i e_i\|$ for all A and B.

If X has a shrinking basis $\{e_i\}$, then it follows from **[6]** that the operators with finite matrices are norm dense in $K(X)$. Hence, in this case, we can associate a matrix to each $f \in K(X)^*$ such that f is determined by its matrix.

LEMMA 2.3. *Let X have an unconditionally monotone, shrinking basis.*

(1) *For each $f \in K(X)^*$, the functional obtained from the matrix of f by replacing with zeros any set of rows or columns will have norm $\leq \|f\|$.*

(2) *If a matrix in $K(X)$ consists of a single nonzero column (row), its norm in $K(X)$ is equal to its norm as an element of X (X^*).*

(3) *If a matrix in $K(X)^*$ consists of a single nonzero column (row), its norm in $K(X)^*$ is equal to its norm as an element of X^* (X^{**}).*

These facts are proved in **[2]**.

DEFINITION 2.4. We shall call a basis $\{e_i\}$ uniformly smooth if, for each $\varepsilon > 0$, $\exists \delta > 0$ such that $\|x + y\| < 1 + \varepsilon\|y\|$ whenever x and y have disjoint supports, $\|x\| = 1$ and $\|y\| < \delta$. We shall call $\{e_i\}$ quasi-uniformly smooth if, for each $\varepsilon > 0$, $\exists \delta > 0$ such that $\|e_i + \lambda e_j\| < 1 + \delta\varepsilon$ for all i, j, whenever $|\lambda| < \delta$. Note that if a basis is uniformly smooth, the Banach space itself need not be uniformly smooth. For example, consider the standard basis for c_0.

The following is a generalization of the Smith-Ward result, since the hypotheses of the theorem are satisfied if X is l_p, $1 < p < \infty$.

THEOREM 2.5. *Let X be a Banach space with an unconditionally monotone, uniformly smooth basis $\{e_i\}$ and with $\{e_i^*\}$ a quasi-uniformly smooth basis for X^*. Then any nontrivial M-ideal in $B(X)$ must contain $K(X)$, and $B(X)$ does not contain any nontrivial M-summands.*

PROOF. Let f_{ij} denote the functional with a one in the ij place and zeros elsewhere. We claim that $[f_{ij}\colon \text{all } i,j] = K(X)^*$. For suppose the contrary, i.e., suppose that there exists an $f \in K(X)^*$ which is not a uniform limit of finite matrix elements of $K(X)^*$. Since $\{e_i\}$ is shrinking, we can assume w.l.g. that $\|f_n\| \downarrow 1$, where f_n is the functional formed from f by deleting the first n rows and columns from the matrix for f. Pick $\delta < 1$ corresponding to $\varepsilon = 1/2$ in the definition of a uniformly smooth basis. Then pick N such that $\|f_N\| < (1 + \frac{3}{4}\delta)/(1 + \frac{1}{2}\delta)$ and choose T and U norm one, disjoint operators (i.e. $\exists m$ such that $t_{ij} = 0$ if i or $j > m$ and $u_{ij} = 0$ if i or $j < m$) with both $f_N(T)$ and $f_N(U) > 1 - \delta/8$. Then,

$$\frac{f_N(T + \delta U)}{\|T + \delta U\|} > \frac{1 + \frac{3}{4}\delta}{1 + \frac{1}{2}\delta},$$

which is a contradiction. Hence, $[f_{ij}\colon \text{all } i,j] = K(X)^*$.

Each f_{ij} must be extreme in the unit ball of $K(X)^*$, for suppose that $f_{ij} + g$ has a one in the ij place and an $\varepsilon > 0$ in the kl place. For this ε, let δ be the smaller of the smoothness δ's for $\{e_i\}$ and $\{e_i^*\}$. Then for T, the operator with $t_{ij} = 1$, $t_{kl} = \delta$, and zeros elsewhere, we have $(f_{ij} + g)T = 1 + \delta\varepsilon$ and $\|T\| < 1 + \delta\varepsilon$.

In **[4]** it was shown that if X has an unconditionally monotone, uniformly smooth basis, then $K(X)$ is an HB-subspace of $B(X)$.

Now suppose that J is a nontrivial M-ideal in $B(X)$. Each f_{kl} is extreme in the unit ball of $K(X)^*$ and hence, by Lemma 1.7, each f_{kl} must be in J_* or

$J^{\perp}$. Let $T \neq 0$ be in J and pick f_{ij}: $f_{ij}(T) \neq 0$. Then f_{ij} must be in J_*. Next suppose that some $f_{mn} \in J^{\perp}$. This would contradict the fact that J is an M-ideal, since $\|f_{ij} + f_{mj}\|$ and $\|f_{mn} + f_{mj}\|$ both have norm less than 2 by Lemma 2.3 and the smoothness hypotheses. Thus, $[f_{ij}$: all $i,j] \subset J_*$ and by Lemma 1.5 $K(X) \subset J$.

$B(X)$ has no nontrivial M-summands since, for each norm one $U \in B(X)$, $\exists$ an operator E_{ij} with a one in the ij place and zeros elsewhere such that $\|U + E_{ij}\| > 1$.

COROLLARY 2.6. *For $X = d(a,p)$, any Lorentz sequence space with $1 < p < \infty$, the hypotheses of Theorem 2.5 are satisfied.*

PROOF. To see that the basis $\{e_i^*\}$ is quasi-uniformly smooth, note that for each $\delta > 0$, $e_i^* + \delta e_j^*$ will achieve its norm on an element of the form $(e_i + \lambda_\delta e_j)/\|e_i + \lambda_\delta e_j\|$, such that $\lambda_\delta \to 0$ as $\delta \to 0$. The basis $\{e_i\}$ is uniformly smooth, since $\|x + y\|^p \leq \|x\|^p + \|y\|^p$, whenever x and y are disjoint.

COROLLARY 2.7. *For each j let X_j be a space with an unconditionally monotone, uniformly smooth basis $\{e_i^j\}$ and a quasi-uniformly smooth basis $\{e_i^{j*}\}$ such that for each $\varepsilon > 0$, there is a common smoothness δ for all j. Then the hypotheses of Theorem 2.5 are satisfied for $(\sum_{j=1}^{\infty} \oplus X_j)_{l_p}$.*

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