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MONOTONE OPERATOR FUNCTIONS ON ARBITRARY SETS

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ABSTRACT. We give a new proof of a result of Chandler which shows that a monotone operator function defined on a set J admits an analytic continuation to the upper and lower half-planes, and that this continuation is a Pick function, real and regular on the convex hull of J.

Let J be a subset of the real axis and $f(x)$ a real-valued function defined on J. If H is a selfadjoint operator on Hilbert space having a spectrum contained in J, the operator $f(H)$ is defined by the usual operational calculus:

$$f(H) = \int f(\lambda)\, dE_\lambda$$

where E_λ is the resolution of the identity corresponding to H. Here $f(H)$ is defined only if $f(x)$ is subject to certain mild measurability conditions. However, in the special case when the Hilbert space is finite dimensional, no measurability conditions need be imposed for $f(H)$ to make sense, and indeed, if H is represented by a diagonal matrix with eigenvalues λ_i, then $f(H)$ is represented by another diagonal matrix with eigenvalues $f(\lambda_i)$.

The function $f(x)$ is said to be a *monotone matrix function of order n* if, for any n-dimensional Hilbert space and any pair of selfadjoint operators A and B on that space having their spectra in J, the operator inequality $A < B$ implies $f(A) < f(B)$. The function $f(x)$ is called a *monotone operator* function if a similar assertion holds for infinite dimensional Hilbert space.

In the special case when the set J is an open interval of the real axis the theory of monotone operator functions and monotone matrix functions is more or less complete. A theorem of Bendat and Sherman [1] asserts that the monotone operator functions defined on the interval J are exactly those functions that are monotone matrix functions of all orders on the interval. A well-known theorem of Loewner [6] then guarantees that these functions are precisely those real functions on the interval J which admit an analytic continuation throughout the upper half-plane having a positive imaginary part in that half-plane. The functions are also continuable by reflection across J to the lower half-plane. Since every Pick function, (i.e., one analytic in the upper half-plane with positive imaginary part) admits a unique canonical representation of the form

$$f(z) = \alpha z + \beta + \int \left[\frac{1}{\lambda - z} - \frac{\lambda}{\lambda^2 + 1} \right] d\mu(\lambda) \tag{1}$$

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where $\alpha \geqslant 0$, β is real and $d\mu$ is a positive measure for which the function $(\lambda^2 + 1)^{-1}$ is integrable, all monotone operator functions on J have such a representation. It is easy to verify that a function of the form (1) is a monotone operator on J if and only if the corresponding measure puts no mass in the open interval J.

In recent years there has been considerable interest in the study of monotone operator functions defined on J when J is an arbitrary open set. Let (a_i, b_i) be the constituent intervals of J. If $f(x)$ is a monotone operator function on J it is surely a monotone operator function on each of the constituent intervals and so admits, from each such interval, an analytic continuation to the upper half-plane. It is tempting to suppose that the continuation so obtained is independent of the interval in question, i.e., that $f(x)$ is of the form (1) where the measure $d\mu$ puts no mass in the open set J. That this is in fact the case is one of the consequences of work of Rosenblum and Rovnyak [7]. Their result has been extended by Chandler [2] who has shown that in fact the measure $d\mu$ puts no mass in the convex hull of J. His results depend on the deep investigations of Rosenblum and Rovnyak as well as results of Šmul'jan [8] which in turn depend on studies of Davis ([3], [4]). We give here a simpler proof of Chandler's theorem in the spirit of the original work of Loewner. Here, if J is a set, $c(J)$ is its convex hull.

CHANDLER'S THEOREM. *A monotone operator function $f(x)$ defined on the open set J is of the form* (1) *where the measure $d\mu$ puts no mass in the open interval $c(J)$. The function is therefore the restriction to J of a monotone operator function on $c(J)$.*

PROOF. We first establish the theorem under the additional hypothesis that J is a bounded set. Let (a', b') and (a'', b'') be two constituent intervals of J; we may suppose that (a', b') is the one on the left. Select a dense sequence $\{x'_k\}$ in (a', b') and another sequence $\{x''_k\}$ dense in (a'', b''). Let ε_n be a sequence of positive numbers converging rapidly to 0 and let $\Psi(x)$ be a monotone operator function defined on $c(J)$ which is not rational. It is easy to find such a function, since any function of the form (1) where the measure has no mass in the interval $c(J)$ and is not a finite collection of point masses will do.

For each positive integer n consider the set

$$S_n = \{x'_1, x'_2, x'_3, \ldots, x'_n, x''_1, x''_2, x''_3, \ldots, x''_n\}$$

consisting of $2n$ points of J and let $\varphi_n(z)$ be a rational function of degree at most $2n$ satisfying the $2n$ equations

$$\varphi_n(x'_i) = f(x'_i) + \varepsilon_n \Psi(x'_i)$$

and

$$\varphi_n(x''_i) = f(x''_i) + \varepsilon_n \Psi(x'_i).$$

That such a function exists and is uniquely determined has been established by Loewner [6]. See also [5]. It is known that $\varphi_n(z)$ is of degree exactly $2n$ and that it has a positive imaginary part in the upper half-plane. It is real on the real axis and has all of its poles there. It is important to notice that these poles are outside the interval (x_1', x_n''). A well-known theorem guarantees that the sequence $\varphi_n(z)$ has a subsequence converging uniformly on compact subsets of the upper half-plane and also uniformly on compact subsets of the interval (a', b''). The limiting function $F(x)$ is manifestly also of the form (1) and coincides with $f(x)$ on the intervals (a', b') and (a'', b''). The associated measure has no mass in the interval (a', b''). Thus $F(x)$ is a monotone operator function when considered on the interval (a', b''). Since the choice of the subintervals (a', b') and (a'', b'') was arbitrary, it is clear that the analytic continuation of $f(x)$ from a constituent interval of J to the half-plane is independent of the choice of the interval, and that the monotone operator function so obtained admits a representation (1) with the associated measure putting no mass in any interval of the form (a_i, b_j) where $a_i < b_j$. This means that $d\mu$ has no mass in the convex hull of J.

In the case when J is not bounded, we consider first $f_N(x)$, the restriction of $f(x)$ to the intersection of J with the interval $(-N, N)$. It is clear that the analytic continuation of $f_N(x)$ to the half-plane is independent of N. It is also clear that the measure $d\mu$ associated with that analytic continuation has no mass in the convex hull of the part of J in $(-N, N)$, and since N is arbitrary, there is no mass in $c(J)$, as asserted. This completes the proof.

It is interesting to note that if $c(J)$ is the whole axis then $f(x)$ is necessarily a linear function with nonnegative slope.

A review of the previous proof makes it clear that we made very little use of the hypothesis that J was an open set. Suppose, for example, that J is an arbitrary set such that $c(J)$ is the open interval (a, b), where we do not exclude the possibility of one or more endpoints at infinity. The construction of the sequence $\varphi_n(z)$ is still possible, selecting the x_k' near a and the x_k'' near b. The existence of the functions $\varphi_n(z)$ depends only on the positivity of certain "Loewner determinants" associated with the points of S_n and this is an algebraic, rather than an analytical fact. See [5]. Thus, as in the proof of Chandler's theorem, we find that when $c(J) = (a, b)$ a monotone operator function defined on J is the restriction to that set of a monotone operator function on $c(J)$.

Circumstances are slightly different when $c(J)$ is closed on the left, say $c(J) = [a, b)$. In this case we form the sequence $\varphi_n(z)$ as before, only always taking x_1' as a, a point of J. The corresponding sequence of Pick functions $\varphi_n(z)$ has a convergent subsequence and the limiting function $F(x)$ coincides with the initial $f(x)$ on J, except, perhaps, at the point a where the inequality will read $F(a) \geqslant f(a)$.

In a similar way we find that if $c(J)$ is of form $(a, b]$ then $f(x)$ coincides on J with a Pick function throughout the interval, except, perhaps, at the right hand end point where the inequality $F(b) \leqslant f(b)$ is valid. It is also obvious how to treat the case when $c(J)$ is a closed interval. We have therefore established the following result.

GENERALIZED CHANDLER THEOREM. *Let J be an arbitrary subset of the real axis and $c(J)$ its convex hull and let $f(x)$ be a monotone operator function defined on J. Then there exists a function $F(x)$ of the form* (1) *so that $F(x) = f(x)$ at all points of J in the interior of $c(J)$ and satisfying the inequalities $F(z) \geqslant f(z)$ and $F(b) \leqslant f(b)$. The function $F(x)$ is associated with a measure $d\mu$ that puts no mass in $c(J)$.*

As a special case of this theorem we obtain a result of Šmul'jan [**8**]: if J consists of a point a and an interval (b, c) where $a < b$ and $f(x)$ is a monotone operator function on J, then $f(x)$ admits an extension $F(x)$ to a moonotone operator function defined on $[a, c)$ which coincides with $f(x)$ on (b, c) and satisfies the inequality $F(a) \geqslant f(a)$.

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