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A MORREY-NIKOL'SKIĬ INEQUALITY

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ABSTRACT. An inequality of Sobolev type is proved which unifies some work of Nikol'skiĭ on fractional derivatives and some work of Morrey which assumes that the growth rate of the L^p norm of the gradient of a function on balls is bounded by some power of the radius.

In this paper we prove the

THEOREM. *Let Q_0 be a cube in R^n and $u(x) \in L^p(Q_0)$. Suppose that*

$$\int_Q |u(x+t) - u(x)|^p \, dx \leqslant K^p |t|^{\alpha p} |Q|^{1-\beta/n} \tag{1}$$

for all parallel subcubes, Q, of Q_0 and for all t such that the integral is defined. The constants are assumed to satisfy $K > 0$, $0 < \alpha \leqslant 1$, $0 \leqslant \beta \leqslant n$, and $p > 1$. Then:

(i) *If $\alpha p < \beta$, then the function $u(x)$ is in $L^r(Q_0)$ for all r satisfying $1/r > 1/p - \alpha/\beta$ and*

$$\left(\int_{Q_0} |u(x) - u_{Q_0}|^r \, dx \right)^{1/r} \leqslant CK$$

where u_{Q_0} is the average value of $u(x)$ on Q_0.

(ii) *If $\alpha p > \beta$, then the function $u(x)$ is Hölder continuous in Q_0 with exponent $\bar{\alpha} = \alpha - \beta/p$ and*

$$|u(y) - u(x)| \leqslant CK|x - y|^{\bar{\alpha}} \quad \text{for all } x,y \in Q_0.$$

The constant, C, depends only on $|Q_0|$, α, β, p, n and in case (i) *on r. Here $|Q|$ is the volume of the cube Q.*

This result is related to the work of Morrey [5] where the condition

$$\int_{B(x_0, R)} |u_x|^p \, dx \leqslant K^p |B(x_0, r)|^{1-\beta/n}$$

is assumed to hold for all balls, $B(x_0, R)$, in the region considered and to the work of Nikol'skiĭ [6], where the condition

$$\int_{R^n} |u(x+t) - u(x)|^p \, dx \leqslant K^p |t|^{\alpha p}$$

is studied. Actually the closest result to ours in the literature is Stampacchia [7, Theorem 3.2]. If, in case (i), we could weaken our hypothesis to include the possibility that $1/r = 1/p - \alpha/\beta$, we would have a new proof of Nikol'skiĭ's

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results for a cube. Of course Nikol'skiĭ considers primarily the case R^n. Our result may be thought of as a unification of these works.

Our method of proof is a direct attack in the spirit of and inspired by the proof of Lemma 2 in John and Nirenberg [3]. The proof of case (ii) is little more than a paraphrase of their proof–plus an application of a result due to Meyers [4] and Campanato [2]. Meyers' proof uses the technique of Lemma 1 in [3]. The proof of case (i) seems to be considerably different from the work in [3]. In particular, no covering lemma is used. Besides the intrinsic value of our theorem, estimates like (1) can occur in the study of elliptic regularity. We will apply our result to this problem in a future paper.

PROOF OF CASE (i): $\alpha p > \beta$. We will need the

LEMMA. *Let a, b_i, $i = 1, \ldots, k$ and $r > 1$ be real numbers. Then there exist constants $c_1, c_2 > 0$ which are independent of a and the b_i's such that*

$$\sum_{i=1}^{k} \left(|a + b_i|^r - |a|^r\right) \leq c_1 \sum_{i=1}^{k} |b_i|^r + c_2|a|^{r-1} \sum_{i=1}^{k} |b_i|.$$

PROOF. Let $N > 1$. If $|a/b_i| \leq N$, then

$$|a + b_i|^r - |a|^r = |b_i|^r\left(|(a/b_i) + 1|^r - |a/b_i|^r\right) \leq c_2(N)|b_i|^r.$$

If $|a/b_i| > N$, then

$$\begin{aligned} |a + b_i|^r - |a|^r &= |a|^r|(1 + b_i/a)^r - 1| \\ &= |a|^r\left|rb_i/a + r(r-1)/2(b_i/a)^2 + \cdots\right| \leq |a|^r c_1(N)|b_i/a|. \end{aligned}$$

Thus, $|a + b_i|^r - |a|^r \leq c_1|a|^{r-1}|b_i| + c_2|b_i|^r$. Adding, we obtain the result.

We will assume that $|Q_0| = 1$. The general case follows by a simple change of scale. We now develop an appropriate representation for

$$\int_{Q_0} |u(x) - u_{Q_0}|^r \, dx.$$

Let $N \geq 1$ be an integer. We subdivide Q_0 into 2^{nN} equal subcubes. The subcubes are denoted by $Q_{N,i}$, $i = 1, \ldots, 2^{nN}$. The indexing of the subcubes is done in such a way that the subcubes $Q_{N,i}$, $i = (k-1)2^n + 1, \ldots, k2^n$; are contained in the subcube $Q_{N-1,k}$ for $N = 2, 3, \ldots,$ and for $k = 1, \ldots, 2^{n(N-1)}$. Let $u_{N,i}$ be defined by

$$u_{N,i} = (1/|Q_{N,i}|)\int_{Q_{N,i}} \left(u(x) - u_{Q_0}\right) dx.$$

Let $A_N = 2^{-nN}\sum_{i=1}^{2^{nN}}|u_{N,i}|^r$. It is easy to see that

$$\lim_{N\to\infty} A_N = \int_{Q_0|u(x) - u_{Q_0}|^r} dx$$

in the sense that, if either side is finite, so is the other and they are equal. Let $a_{N,i}$ be defined according to the formula

$$a_{N,(k-1)2^n+j} = u_{N,(k-1)2^n+j} - u_{N-1,k}$$

where $N = 2, 3, \dots, k = 1, \dots, 2^{n(N-1)}$, and $j = 1, \dots, 2^n$. Let $a_{1,i} \equiv u_{1,i}$, $i = 1, \dots, 2^n$. Finally, let $b_{N,k,j} \equiv a_{N,(k-1)2^n+j}$. Using this notation we have

$$A_N = 2^{-nN} \sum_{k=1}^{2^{n(N-1)}} \left(\sum_{j=1}^{2^n} \left| u_{N-1,k} + b_{N,k,j} \right|^r \right).$$

Now,

$$\begin{aligned} 0 &\leqslant A_N - A_{N-1} \\ &= 2^{-nN} \sum_{k=1}^{2^{n(N-1)}} \left(\sum_{j=1}^{2^n} \left| u_{N-1,k} + b_{N,K,j} \right|^r - \left| u_{N-1,k} \right|^r \right) \\ &\leqslant 2^{-nN} \sum_{k=1}^{2^{n(N-1)}} \left(\sum_{j=1}^{2^n} c_1 \left| b_{N,k,j} \right|^r + c_2 \left| u_{N-1,k} \right|^{r-1} \left| b_{N,k,j} \right| \right), \end{aligned}$$

(here c_1 and c_2 are constants from our lemma)

$$\begin{aligned} &\leqslant 2^{-nN} \left[c_1 \sum_{j=1}^{2^{nN}} \left| a_{N,j} \right|^r + c_2 2^{n(1-1/r)} \sum_{k=1}^{2^{n(N-1)}} \left| u_{N-1,k} \right|^{r-1} \left(\sum_{j=1}^{2^n} \left| b_{N,k,j} \right|^r \right)^{1/r} \right] \\ &\leqslant 2^{-nN} \left[c_1 \sum_{j=1}^{2^{nN}} \left| a_{N,j} \right|^r + c_2 2^{n(1-1/r)} \left(A_{N-1} 2^{n(N-1)} \right)^{1-1/r} \left(\sum_{j=1}^{2^{nN}} \left| a_{N,j} \right|^r \right)^{1/r} \right] \\ &= c_1 X_N + c_2 A_{N-1}^{1-1/r} X_N^{1/r}, \quad \text{where } X_N \equiv 2^{-nN} \sum_{j=1}^{2^{nN}} \left| a_{N,j} \right|^r. \end{aligned}$$

Suppose that

$$X_N \leqslant c y^N \tag{2}$$

where c is a constant independent of N and $0 \leqslant y < 1$. Then $A_N \leqslant A_{N-1} + c_1 y^N + c_2 A_{N-1}^{1-1/r} y^{N/r}$, $N = 2, 3, \dots$. Let $B_1 = \max(1, A_1)$. Then the sequence defined by

$$B_N = B_{N-1}\left(1 + ((c_1 + c_2 B_{N-1})/B_{N-1}) y^{N/r}\right), \qquad N = 2, 3, \dots,$$

majorizes the sequence $\{A_N\}$. Since $B_N \geqslant 1$ for all N, $(c_1 + c_2 B_{N-1})/B_{N-1} \leqslant d < \infty$ and d is independent of N. The sequence C_N defined by

$$\begin{aligned} C_N &= C_{N-1}\left(1 + d y^{N/r}\right), \qquad N = 2, 3, \dots, \\ C_1 &= B_1, \end{aligned}$$

majorizes the sequence B_N. By elementary results on infinite products the sequence C_N converges and of course we can get explicit estimates. All that remains, therefore, is to prove (2).

To this end, we establish two estimates: a "*global*" *estimate* and a "*local*" *estimate*.

The Nth "global" estimate. There are 2^n vertices V_i, $i = 1, \dots, 2^n$, of Q_0. For each pair of vertices (V_i, V_j) there correspond two cubes Q_i and Q_j which satisfy $Q_i, Q_j \subset Q_0$, V_i and V_j are vertices of Q_i and Q_j respectively and the edge length of Q_i and Q_j is $1 - 2^{-N}$. Choose $t_{i,j}$ to translate Q_i to Q_j. The cube

Q_i is composed of $(2^N - 1)^n$ of the cubes $Q_{N,i}$, $i = 1, \ldots, 2^{nN}$. Rename these cubes $Q_{i,k}$, $k = 1, \ldots, (2^N - 1)^n$.

We have

$$\begin{aligned}
|Q_{i'} \cdot|^{1-p} &\sum_{k=1}^{(2^N-1)^n} \left| \int_{\bar{Q}_{i,k}} u(x)\, dx - \int_{Q_{i,k}} u(x)\, dx \right|^p \\
&= |Q_{i'} \cdot|^{1-p} \sum_{k=1}^{(2^N-1)^n} \left| \int_{Q_{i,k}} u(x + t_{i,j}) - u(x)\, dx \right|^p \\
&\leqslant \sum_{k=1}^{(2^N-1)^n} \int_{Q_{i,k}} |u(x + t_{i,j}) - u(x)|^p\, dx \\
&= \int_{Q_i} |u(x + t_{i,j}) - u(x)|^p\, dx < K^p n^{\alpha p/2} 2^{-\alpha N p}.
\end{aligned}$$

Here $\bar{Q}_{i,j}$ is the translation of $Q_{i,j}$ by $t_{i,j}$. Summing these inequalities corresponding to all possible pairs of vertices and discarding irrelevant terms from the left-hand side we obtain

$$\sum |a_{N,i} - a_{N,j}|^p \leqslant K^p n^{\alpha p/2} \binom{2^n}{2} 2^{N(n-\alpha p)}$$

where the sum is over all i, j such that $Q_{N,i}$ and $Q_{N,j}$ belong to the same subcube in the $(N - 1)$st partition of Q_0.

Since $\sum_{j=1}^{2^n} b_{N,k,j} = 0$, one can easily show that

$$\sum_{j=1}^{2^n} |b_{N,k,j}|^p \leqslant \sum_{1 < i < j < 2^n} |b_{N,k,i} - b_{N,k,j}|^p.$$

Hence

$$\sum_{j=1}^{2^{nN}} |a_{N,j}|^p \leqslant K^p n^{\alpha p/2} \binom{2^n}{2} 2^{N(n-\alpha p)}. \tag{3}$$

The Nth "local" estimate. Select two subcubes from the Nth subdivision belonging to the same cube from the $(N - 1)$st subdivision. Choose t to translate one of the cubes to the other. Fixing the $(N - 1)$st subcube and summing the inequalities resulting from applying (1), to all possible pairs of Nth subdivision cubes, we get

$$\sum_{j=1}^{2^n} |b_{N,k,j}|^p \leqslant K^p n^{\alpha p/2} \binom{2^n}{2} 2^{N(\beta-\alpha p)}.$$

So in particular

$$|a_{N,j}|^p \leqslant K^p n^{\alpha p/2} \binom{2^n}{2} 2^{N(\beta-\alpha p)}, \qquad j = 1, \ldots, 2^{nN}. \tag{4}$$

We now obtain an upper bound for

$$\sum_{j=1}^{2^{nN}} |a_{N,j}|^r \tag{5}$$

subject to the conditions (3) and (4). With these restrictions, it is easy to see that to make (5) as large as possible one of its terms should be made as large as possible, then another should be made as large as possible, etc. Since the largest $|a_{N,j}|^p$ can be is given by (4) and because of (3), the number of these largest terms is bounded by $2^{N(n-\beta)}$. All the other terms must be zero. This leads to the estimate

$$\sum_{j=1}^{2^{nN}} |a_{N,j}|^r \leq K^r n^{r/2} \binom{2^n}{2}^{r/p} 2^{N(n-\beta+r(\beta-\alpha p)/p)}.$$

Thus $X_N \leq C 2^{(-\beta+r(\beta-\alpha p)/p)N} = cy^N$. But since $-\beta + r(\beta - \alpha p)/p < 0$ by hypothesis, $y < 1$. This completes the proof.

Proof of the case $\alpha p > \beta$. Consider a subcube Q of Q_0 of edge length e. Subdivide Q into 2^{nN} equal subcubes, Q_i, $i = 1, \ldots, 2^{nN}$. Let u_i denote the average value of u on Q_i. Then

$$\frac{1}{|Q|}\int_Q |u - u_Q|\, dx = \lim_{N\to\infty} 2^{-nN} \sum_{i=1}^{2^{nN}} |u_i - u_Q|.$$

By the Schwarz inequality,

$$2^{-nN} \sum_{i=1}^{2^{nN}} |u_i - u_Q| \leq \left[2^{-nN} \sum_{i=1}^{2^{nN}} |u_i - u_Q|^2\right]^{1/2} \equiv A_N^{1/2},$$

where the A_N are so defined. Following John and Nirenberg we conclude that

$$2^{2nN} A_N = \tfrac{1}{2} \sum_{i,j=1}^{2^{nN}} |u_i - u_j|^2.$$

Now subdivide Q into $2^{n(N+1)}$ equal subcubes. Each of the subcubes Q_i is thereby divided into 2^n cubes $Q_{i,j}$, $j = 1, \ldots, 2^n$, $i = 1, \ldots, 2^{nN}$. And $u_i = 2^{-n}\sum_{j=1}^{2^n} u_{i,j}$, where $u_{i,j}$ is the average values of u on $Q_{i,j}$.

We next note that

$$\left|\int_{Q_{i,j}} u(x+t) - u(x)\, dx\right| \leq \left(\int_{Q_{i,j}} |u(x+t) - u(x)|^p\, dx\right)^{1/p} |Q_{i,j}|^{1-1/p}$$
$$\leq K|Q_{i,j}|^{(1-\beta/n)/p}|Q_{i,j}|^{1-1/p}|t|^\alpha = K|Q_{i,j}|^{1-\beta/pn}|t|^\alpha.$$

Choose t so that $\int_{Q_{i,j}} u(x+t)\, dx = \int_{Q_{i,k}} u(x)\, dx$ where $Q_{i,k}$ and $Q_{i,j}$ have one face in common. Thus for any two subcubes $Q_{i,j}$ and $Q_{i,k}$ of Q_i,

$$|u_{i,j} - u_{i,k}| \leq Knh^{\alpha-\beta/p} = M_{N+1}$$

where $h = e2^{-N}$.

Following John and Nirenberg, it can be shown that $A_{N+1} \leq A_N + M_{N+1}^2$, $N = 0, 1, 2, \ldots$. Now, since $A_0 = 0$,

$$A_N \leq \sum_{j=1}^N M_j^2 = K^2 n^2 \sum_{j=1}^N \left(\frac{e^2}{2^{2j}}\right)^{\alpha-\beta/p} = K^2 n^2 e^{2(\alpha-\beta/p)} \sum_{j=1}^N x^j,$$

where $x = 2^{-2(\alpha-\beta/p)} < 1$.

Hence, $A_N \leqslant Ce^{2(\alpha-\beta/p)}$ and

$$\frac{1}{|Q|}\int_Q |u-u_Q|\,dx \leqslant C|Q|^{(\alpha-\beta/p)/n}.$$

The result now follows from Meyers [4] or Campanato [2].

Remark 1. It is not hard to prove more general versions of case (ii). For example, if $n = 1$ and $\phi_1(x)$ and $\phi_2(x)$ are increasing functions and if

$$\left(\int_Q |u(x+t)-u(x)|^p\,dx\right)^{1/p} \leqslant K\phi_1(|t|)\phi_2(|Q|),$$

then

$$(1/|Q|)\int_Q |u-u_Q|\,dx \leqslant \left(\int_0^{|Q|}\phi_1^2(x)\phi_2^2(x)x^{-1-2/p}\,dx\right)^{1/2} \equiv g(|Q|).$$

In particular, if $g(|Q|)\to 0$, we have that $u(x)\in \mathrm{VMO}(Q_0)$, the space of functions of vanishing mean oscillation.

Remark 2. Before discovering our proof, we had tried several approaches which did not work. The method of Stampacchia [7] does not work and the method of symmetrization used so successfully by Garsia and Rodemich in [2] and elsewhere does not work because the appropriate functional is not decreased by symmetrization.

Bibliography

1. A. Garsia and E. Rodemich, *Monotonicity of certain functionals under rearrangement*, Ann. Inst. Fourier (Grenoble) **24** (1974), 67–116.

2. S. Campanato, *Propieta di una famiglia di Spazi funcionalli*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. **18** (1964), 137–160.

3. F. John and L. Nirenberg, *On functions of bounded mean oscillation*, Comm. Pure Appl. Math. **14** (1961), 415–426.

4. N. Meyers, *Mean oscillation over cubes and Hölder continuity*, Proc. Amer. Math. Soc. **15** (1964), 717–721.

5. C. B. Morrey, *On the solution of quasi-linear elliptic partial differential equations*, Trans Amer. Math. Soc. **43** (1938), 126–166.

6. S. M. Nikol'skiĭ, *Approximation of functions of several variables and imbedding theorems*, Springer-Verlag, Berlin and New York, 1975.

7. G. Stampacchia, *The spaces $L^{(p,\lambda)}$, $N^{(p,\lambda)}$ and interpolation*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. **19** (1965), 293–306.

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