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THERE ARE NO Q-POINTS IN LAVER'S MODEL FOR THE BOREL CONJECTURE

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ABSTRACT. It is shown that it is consistent with ZFC that no nonprincipal ultrafilter on ω is a Q-point (also called a rare ultrafilter).

All ultrafilters are assumed to be nonprincipal and on ω.

DEFINITIONS. (1) U is a Q-point (also called rare [C]) iff $\forall f \in \omega^\omega$ if f is finite-to-one then $\exists X \in U, f \restriction X$ is one-to-one.

(2) U is a P-point iff $\forall f \in \omega^\omega$, $\exists X \in U, f \restriction X$ is constant or finite-to-one.

(3) U is a semi-Q-point (also called rapid [C], iff $\forall f \in \omega^\omega, \exists g \in \omega^\omega, \forall n\, f(n) < g(n)$ and $g''\omega \in U$.

(4) U is semiselective iff it is a P-point and a semi-Q-point.

(5) For $f, g \in \omega^\omega$, $[f < g$ iff $\exists n\, \forall m > n\, (f(m) < g(m))]$.

(6) For $\mathfrak{F} \subseteq \omega^\omega$, $[\mathfrak{F}$ is dominant iff $\forall f \in \omega^\omega\, \exists g \in \mathfrak{F}\, (f < g)]$.

THEOREM 1 (KETONEN [Ke]). *If every dominant family has cardinality* $2^{\aleph_0}$, *then there exists a P-point.*

THEOREM 2 (MATHIAS, TAYLOR [M3]). *If there exists a dominant family of cardinality* $\aleph_1$, *then there exists a Q-point.*

Kunen [Ku1] showed that adding $\aleph_2$ random reals to a model of ZFC + GCH gives a model with no semiselective ultrafilters. More recently he showed [Ku2] that if one first adds $\aleph_1$ Cohen reals (then the random reals) then the resulting model has a P-point. In either case one has a dominant family of size $\aleph_1$ so there is a Q-point.

THEOREM 3. *The following are equivalent*:

(1) *U is a semi-Q-point.*

(2) *Given $P_n \subseteq \omega$ finite for $n < \omega$ there exists $X \in U$ such that $\forall n$, $|X \cap P_n| \leq n$.*

(3) *$\exists h \in \omega^\omega$ such that given $P_n \subseteq \omega$ finite for $n < \omega$ there exists $X \in U$ such that $\forall n$, $|X \cap P_n| \leq h(n)$.*

PROOF. (1) $\Rightarrow$ (2). Let $f(n) = \sup(\bigcup_{m \leq n} P_m) + 1$. Suppose that for all n, $g(n) > f(n)$; then $P_n \cap g''\omega \subseteq \{g(0), \ldots, g(n-1)\}$.

(3) $\Rightarrow$ (1). Assume f increasing. Choose $n_0 < n_1 < n_2 < \cdots$, so that $h(k+1) < n_k$. Let $P_k = f(n_k)$ and let $Y \in U$ so that $|Y \cap P_k| \leq h(k)$. Then, for each $m \geq n_0$, $|Y \cap f(m)| < m$, since if $n_k \leq m < n_{k+1}$ then

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$$|Y \cap f(n_{k+1})| \leqslant h(k+1) < n_k \leqslant m.$$

Hence if $g \in \omega^\omega$ enumerates $Y - f(n_0 + 1)$ in increasing order then $\forall n$, $f(n) < g(n)$. $\square$

Define $U \times V = \{A \subseteq \omega \times \omega\colon \{n\colon \{m\colon (n, m) \in A\} \in V\} \in U\}$. Whilst $U \times V$ is never a P-point or a Q-point, nevertheless:

THEOREM 4. *$U \times V$ is a semi-Q-point iff V is a semi-Q-point.*

PROOF. ($\Rightarrow$) Given $P_k \subseteq \omega$ finite let $P_k^* = \{\langle n, m\rangle\colon m \in P_k$ and $n \leqslant m\}$. Choose $Z \in U \times V$ so that $\forall k, |Z \cap P_k^*| \leqslant k$. Let $n \in \omega$ so that $Y = \{m \geqslant n\colon (n, m) \in Z\} \in V$ then $\forall k, |Y \cap P_k| \leqslant k$. (More generally if $f_* U = V$ and U is a semi-Q-point and f is finite-to-one then V is a semi-Q-point.)

($\Leftarrow$) Given $P_k \subseteq \omega^2$ finite, choose n_k increasing so that $P_k \subseteq n_k^2$. Let $Y \in V$ so that $\forall k$, $|n_k \cap Y| \leqslant k$. Let $Z = \bigcup_{k<\omega}\{k\} \times \{m\colon m \in Y$ and $m > n_k\}$ then

$$Z \cap P_k \subseteq Z \cap n_k^2 \subseteq k \times (n_k \cap Y)$$

which has cardinality $\leqslant (k+1)^2$. $\square$

THEOREM 5. *In Laver's model N for the Borel conjecture* [L] *there are no semi-Q-points.*

PROOF. Some definitions from [L]:

(1) $T \in \mathcal{T}$ iff T is a subtree of $\omega^{<\omega}$ with the property that there exist $s \in T$ (called stem T) so that $\forall t \in T$, $t \subseteq s$ or $s \subseteq t$, and if $t \supseteq s$ and $t \in T$ then there are infinitely many $n \in \omega$ such that $t^\frown\langle n\rangle \in T$.

(2) $\hat{T} \geqslant T$ iff $\hat{T} \subseteq T$.

(3) $T_s = \{t \in T\colon s \subseteq t$ or $t \subseteq s\}$.

(4) $T^0 \geqslant \hat{T}$ iff $T \geqslant \hat{T}$ and they have the same stem.

(5) For $x < y < \omega$ let $[x, y) = \{n < \omega\colon x \leqslant n < y\}$.

LEMMA 1. *Suppose we are given $T \in \mathcal{T}$ and finite sets F_s for each $s \in T - \{\varnothing\}$ such that for each $s \in T - \{\varnothing\}$:*

(a) *if $s = (k_0, \ldots, k_n, k_{n+1})$, then $F_s \subseteq [k_n, k_{n+1})$;*

(b) *if $s = \langle n\rangle$, then $F_s \subseteq [0, n)$;*

(c) *$\exists N < \omega$ $\forall t$ immediately below s in $T |F_t| \leqslant N$. For any $\hat{T} \geqslant T$ let $H_{\hat{T}} = \bigcup\{F_s\colon s \in \hat{T}\}$. Then $\exists T^1, T^0 \geqslant T$ such that $H_{T^0} \cap H_{T^1}$ is finite.*

PROOF. We may as well assume that the stem of T is $\varnothing$. Given Q any infinite family of sets of cardinality $< N < \omega$ there exists G, $|G| < N$, $\exists \hat{Q} \subseteq Q$ infinite so that $\forall F, \hat{F} \in \hat{Q}$, $F \cap \hat{F} \subseteq G$ (i.e., a Δ-system). Now trim T to obtain $\hat{T} \geqslant T$ so that $\forall s \in T$, $\exists G_s \subseteq [k_n, \omega]$ finite ($s = (k_0, \ldots, k_n)$) and for all t, $\hat{t}$ immediately below s in $\hat{T}$, $(F_t \cap F_{\hat{t}}) \subseteq G_s$. Build two sequences of finite subtrees of $\hat{T}$:

$$T_n^0 \subseteq T_{n+1}^0 \cdots, \qquad T_n^1 \subseteq T_{n+1}^1 \cdots$$

so that

$$\left[\bigcup_{s\in T_n^0}(F_s\cup G_s)\right]\cap\left[\bigcup_{s\in T_n^1}(F_s\cup G_s)\right]\subseteq G_\varnothing$$

and $\bigcup_{n<\omega}T_n^i = T^i \geqslant \hat{T}$ for $i = 0, 1$.

This is done as follows: Suppose we have T_n^0, T_n^1 and we are presented with $s \in T_n^0$ and asked to add an immediate extension of s to T_n^0. Then since $\{F_t - G_s\colon t$ immediately below s in $\hat{T}\}$ is a family of disjoint sets and $G_t \subseteq [k_n, \omega]$ where $t = (k_0, \ldots, k_n)$ we can find infinitely many t immediately below s in $\hat{T}$ so that

$$[(F_t - G_s)\cup G_t]\cap\left[\bigcup_{s\in T_n^1}(F_s\cup G_s)\right] = \varnothing. \quad \square$$

The above is a double fusion argument.

Some more definitions from **[L]**:

(1) Fix a natural ω-ordering of $\omega^{<\omega}$ and for any $T \in \mathfrak{F}$ transfer it to $\{t \in T\colon \text{stem } T \subseteq t\}$ in a canonical fashion. $T\langle n\rangle$ denotes the nth element of $\{t \in T\colon \text{stem } T \subseteq t\}$.

(2) $\hat{T}\ {}^n\!\geqslant T$ iff $\hat{T} \geqslant T$ and $\forall i \geqslant n$, $\hat{T}\langle i\rangle = T\langle i\rangle$.

(3) The p.o. $\mathbf{P}_{\omega_2}$ is the ω_2 iteration of $\mathfrak{F}$ with countable support ($p\restriction_\alpha \Vdash$ "$p(\alpha) \in \mathfrak{F}^{M[G_\alpha]}$" for all α and $\text{supp}(p) = \{\alpha\colon p(\alpha) \neq \omega^{<\omega}\}$ is countable).

(4) For K finite and $n < \omega$, $p\,{}^n_K\!\geqslant q$ iff [$p \geqslant q$ and $\forall\alpha \in K$, $p\restriction_\alpha \Vdash$ "$p(\alpha)\ {}^n\!\geqslant q(\alpha)$"].

LEMMA 2. *Let f be a term denoting the first Laver real and τ any term. If $p \in \mathbf{P}_{\omega_2}$ and $p \Vdash$ "$\tau \in \omega^\omega$, $\forall n$ $(f(n) < \tau(n))$ and τ increasing" then $\exists Z_0$, Z_1 such that $Z_0 \cap Z_1$ is finite and $\exists p_0$, $p_1 \geqslant p$ such that $p_i \Vdash$ "$\tau\omega \subseteq Z_i$" for $i = 0, 1$.*

PROOF. Construct a sequence $p \leqslant^0_{K_0} p_n \leqslant^0_{K_n} p_{n+1}$ so that $\bigcup_{n<\omega}K_n = \bigcup_{n<\omega}\text{supp}(p_n)$ and $0 \in K_0$. Having gotten p_n, let $s = (k_0, \ldots, k_m)$ be $p_n(0)\langle n\rangle$. Fix $t = (k_0, \ldots, k_m, k_{m+1})$ in $p_n(0)$. Then for each $i \leqslant m+1$,

$$p_t = \langle p_n(0)_t \frown p_n\restriction[1, \omega_2)\rangle \Vdash \text{“}\tau(i) \geqslant k_{m+1} \text{ or } \bigvee_{l<k_{m+1}} \tau(i) = l\text{”}.$$

Hence by applying Lemma 6 of [L] $m+2$ many times we can find $q_t\,{}^n_{K_n}\!\geqslant p_t$ and $F_t \subseteq [k_m, k_{m+1}]$ such that $|F_t| \leqslant (m+2)(n+1)^{|K_n|}$ and $q_t \Vdash$ "$\tau''\omega \cap [k_m, k_{m+1}) \subseteq F_t$". (Note $p_t \Vdash$ "$\forall i \geqslant m+1$, $\tau(i) > k_{m+1}$"). Let $p_{n+1}(0) = (p_n(0) - p_n(0)_s) \cup \bigcup\{q_t(0)\colon t$ is immediately below s in $p_n(0)\}$. Let $p_{n+1}[1, \omega_2)$ be a term denoting $q_t\restriction[1, \omega_2)$ if $q_t(0)$ or $p_n\restriction[1, \omega_2)$ if $p_n(0) - \{t\colon s \subseteq t\}$. Hence $p_{n+1}\,{}^n_{K_n}\!\geqslant p_n$. Now let $\hat{p}$ be the fusion of the sequence of p_n (see [**L**, Lemma 5]). Then for each $t \in \hat{p}(0)$ if $t = \langle k_0, \ldots, k_m, k_{m+1}\rangle$ and $t \supseteq \text{stem } \hat{p}(0)$, then $\langle \hat{p}(0)_t \frown \hat{p}\restriction[1, \omega_2)\rangle \Vdash$ "$\tau''\omega \cap [k_n, k_{n+1}) \subseteq F_t$". For $t \in \hat{p}(0)$ and $t \subsetneq \text{stem } \hat{p}(0)$ let $F_t = k_{m+1}$. Applying Lemma 1 obtain T_0, $T_1 \geqslant \hat{p}(0)$, Z_0 and Z_1 such that $Z_0 \cap Z_1$ is finite, and $\langle T_i \frown p\restriction[1, \omega_2]\rangle \Vdash$ "$\tau''\omega \subseteq Z_i$" for $i = 0, 1$. $\square$

PROOF OF THEOREM 5. Suppose $M[G_{\omega_2}] \Vdash$ "U is a semi-Q-point". Applying an argument of Kunen's we get $\alpha < \omega_2$ such that $U \cap M[G_\alpha] \in M[G_\alpha]$. ($M[G_\beta] \Vdash$ "CH" for all $\beta < \omega_2$ so construct using ω_2-c.c., $\alpha_\lambda < \omega_2$ for $\lambda < \omega_1$ so that $\forall x \in M[G_{\alpha_\lambda}] \cap 2^\omega$, $\mathbf{P}_{\alpha_{\lambda+1}}$ decides "$x \in U$". Let $\alpha = \sup \alpha_\lambda$. Note $M[G_\alpha] \cap 2^\omega = \bigcup_{\beta<\alpha} M[G_\beta] \cap 2^\omega$ since $\aleph_1$ is not collapsed.) By [L, Lemma 11] we may assume $U \cap M \in M$. But Lemma 2 clearly implies that for any V ult. in M, $M[G_{\omega_2}] \Vdash$ "no extension of V is a semi-Q-point." □

REMARKS. (1) A similar argument shows that in the model gotten by ω_2 iteration of Mathias forcing with countable support there are no semi-Q-points. In fact, as Mathias later pointed out to me, the appropriate argument needed is an easy generalization of Theorem 6.9 of [M2].

(2) In [M1] Mathias shows $[\omega \to (\omega)^\omega] \Rightarrow$ [There are no rare filters or nonprincipal ultrafilters.]

(3) In neither the Laver or Mathias models are there small dominant families so by Ketenon [Ke] there is a P-point. Also it is easily shown no ultrafilter is generated by fewer then $\aleph_2$ sets.

(4) Not long after the results of this paper were obtained, Shelah showed that it is consistent that no P-points exist [W]. In his model there is a dominant family of size $\aleph_1$, so there are Q-points. It remains open whether or not it is consistent that there are no P-points or Q-points.

CONJECTURE. Borel conjecture $\Leftrightarrow$ there does not exist a semi-Q-point.

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