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ON SUBMETRIZABILITY AND G_δ-DIAGONALS

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ABSTRACT. Open perfect maps and open compact maps with the property that the collection of all nontrivial fibers is discrete are shown to preserve submetrizability and if $f: X \to Y$ is an open compact or perfect map with the above property such that the union of all nontrivial fibers is G_δ and if X has a $\overline{G_\delta(1)}$-diagonal, then Y is shown to have a G_δ-diagonal.

1. Introduction and terminology. A space X is said to be *submetrizable* if there exists a one-to-one continuous map from X onto a metric space. A space X is said to have a *regular G_δ-diagonal* (*G_δ-diagonal*) if the diagonal set of $X \times X$ is a regular G_δ-set (G_δ-set). According to Zenor's covering characterization [6], a space X has a regular G_δ-diagonal iff there exists a sequence $\{\mathcal{U}_n : n \in N\}$ of open covers of X such that if p, q are distinct points in X, then there exist open neighborhoods U, V of p, q, respectively and an integer $n = n(p, q)$ such that $S(U, \mathcal{U}_n) \cap V = \varnothing$. A space X is said to have a $G_\delta(n)$ ($\overline{G_\delta(n)}$)*-diagonal* if there exists a sequence $\{\mathcal{U}_n : n \in N\}$ of open covers of X such that $\{p\} = \bigcap\{S^n(p, \mathcal{U}_m) : m \in N\}$ ($\{p\} = \bigcap\{\overline{S^n(p, \mathcal{U}_m)} : m \in N\}$) for every point $p \in X$. Obviously a space X has a $G_\delta(1)$-diagonal iff X has a G_δ-diagonal.

In this paper, the author considers the permanence properties of submetrizability and G_δ-diagonals by maps. In the sequel all maps are assumed to be continuous and N always denotes the set of all positive integers.

2. Permanence properties of submetrizability and G_δ-diagonals.

LEMMA 1. *Let f be a one-to-one map from X onto Y. Then*:

(i) *If Y is submetrizable, then so is X.*

(ii) *If Y has a regular G_δ-diagonal, then so does X.*

(iii) *If Y has a $G_\delta(n)$-diagonal, then so does X, where $n \in N$.*

The proofs are trivial and omitted.

Let $\mathcal{C}(X)$ be the hyperspace consisting of all nonempty compact subsets of X with the finite topology in the sense of [4].

LEMMA 2. (i) *If X is submetrizable, then so is $\mathcal{C}(X)$.*

(ii) *If X has a regular G_δ-diagonal, then so does $\mathcal{C}(X)$.*

(iii) *If X has a $\overline{G_\delta(n)}$-diagonal, then $\mathcal{C}(X)$ has a $G_\delta(n)$-diagonal, where $n \in N$.*

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PROOF. (i) Let f: $X \to T$ be a one-to-one map from X onto a metric space T. Then g: $\mathcal{C}(X) \to \mathcal{C}(T)$ defined by $g(K) = f(K)$ for every $K \in \mathcal{C}(X)$ is a one-to-one map onto a metrizable space $\mathcal{C}(T)$ by [4, 4.9.13]. (ii) Let $\{\mathcal{U}_n: n \in N\}$ be a sequence of open covers of X with $\mathcal{U}_{n+1} < \mathcal{U}_n$ for every $n \in N$ satisfying the covering characterization of a regular G_δ-diagonal. Let

$$\langle \mathcal{U}_n \rangle = \{\langle U_1, \ldots, U_k \rangle: U_1, \ldots, U_k \in \mathcal{U}_n, k \in N\},$$

$$\langle U_1, \ldots, U_k \rangle = \left\{ K \in \mathcal{C}(X): K \subset \bigcup_{i=1}^{k} U_i, K \cap U_i \neq \varnothing \right.$$

$$\left. \text{for every } i = 1, \ldots, k \right\}.$$

To see that $\{\langle \mathcal{U}_n \rangle: n \in N\}$ forms a regular G_δ-diagonal, let K_1, K_2 be distinct points in $\mathcal{C}(X)$. Without loss of generality, take a point $p \in K_1 - K_2$. For each $q \in K_2$, there exist open neighborhoods $V(p, q)$, $W(q)$ of p, q, respectively, and $n(q) \in N$ such that $V(p, q) \cap S(W(q), \mathcal{U}_{n(q)}) = \varnothing$. Let $\{W(q_j): j = 1, \ldots, k\}$ be a finite subcover of K_2 of $\{W(q): q \in K_2\}$. Put

$$\mathcal{W} = \langle W(q_1), \ldots, W(q_k) \rangle,$$
$$n = \max\{n(q_1), \ldots, n(q_k)\},$$
$$V(p) = \bigcap_{j=1}^{k} V(p, q_j).$$

Then $V(p)$ is an open neighborhood of p such that

$$S(V(p), \mathcal{U}_n) \cap \bigcup_{j=1}^{k} W(q_j) = \varnothing.$$

Construct an open neighborhood $\mathcal{V} = \langle V(p), X \rangle$ of K_1. $\mathcal{V}$ satisfies $S(\mathcal{V}, \langle \mathcal{U}_n \rangle) \cap \mathcal{W} = \varnothing$. Hence $\mathcal{C}(X)$ has a regular G_δ-diagonal. (iii) Let $\{\mathcal{U}_m: m \in N\}$ be a sequence of open covers of X with $\mathcal{U}_{n+1} < \mathcal{U}_n$ for every n satisfying the conditions of a $\overline{G_\delta(n)}$-diagonal. Suppose $p \in K_1 - K_2$, where K_1, K_2 are distinct points in $\mathcal{C}(X)$. Then it follows easily that $S^n(p, \mathcal{U}_m) \cap K_2 = \varnothing$ for some $m \in N$. Then we have by a routine check $K_2 \notin S^n(K_1, \langle \mathcal{U}_m \rangle)$, proving that $\mathcal{C}(X)$ has a $G_\delta(n)$-diagonal.

THEOREM 1. *Let f be an open perfect map from X onto Y. Then*:

(i) *If X is submetrizable, then so is $\mathcal{C}(Y)$.*

(ii) *If X has a regular G_δ-diagonal, then so does $\mathcal{C}(Y)$.*

(iii) *If X has a $\overline{G_\delta(n)}$-diagonal, then $\mathcal{C}(Y)$ has a $G_\delta(n)$-diagonal.*

PROOF. Define g: $\mathcal{C}(Y) \to \mathcal{C}(X)$ by $g(K) = f^{-1}(K)$ for every $K \in \mathcal{C}(Y)$. Then g is one-to-one. To see the continuity of g, let $g(K) \in \langle V_1, \ldots, V_k \rangle = \mathcal{V}$, where $V_1, \ldots, V_k$ are open in X. Put $W = Y - f(X - V_1 \cup \cdots \cup V_k)$ and $\mathcal{W} = \langle W \cap f(V_1), \ldots, W \cap f(V_k) \rangle$. Then $\mathcal{W}$ is an open neighborhood of K such that $g(\mathcal{W}) \subset \mathcal{V}$. Thus g is a one-to-one continuous map from $\mathcal{C}(Y)$

onto the subspace of $\mathcal{C}(X)$. Taking Lemmas 1 and 2 into account, we can conclude (i), (ii) and (iii).

An example due to Popov [5] shows that (i), (ii) and (iii) do not hold if the openness of f is dropped. Since Y is homeomorphic to the subspace of $\mathcal{C}(Y)$, we can say the following remark and this gives an affirmative answer to the question "whether open perfect maps preserve submetrizability or not?" which is proposed by Martin [3].

REMARK. (i) Open perfect maps preserve submetrizability and regular G_δ-diagonals. (ii) If Y is an open perfect image from a space with a $\overline{G_\delta(n)}$-diagonal, then Y has a $G_\delta(n)$-diagonal.

Martin also proposed in [3] the following question: If f is a perfect map from a submetrizable space X onto Y such that

the family of all nontrivial fibers is discrete in X, (*)

then must Y be submetrizable?

If we exchange the closedness of f with openness, then we have a positive result.

THEOREM 2. *Let f: $X \to Y$ be an open compact map with (*). If X is submetrizable, then so is Y.*

PROOF. Define g: $Y \to \mathcal{C}(X)$ as follows: $g(y) = f^{-1}(y)$ for every $y \in Y$. To see the continuity of g, suppose $g(y) = f^{-1}(y) \in \langle V_1, \dots, V_k\rangle \cap g(Y) = \mathcal{V}$, where $V_1, \dots, V_k$ are open sets in X. Put $Y_1 = \{y \in Y: f^{-1}(y)$ is non-trivial, i.e., $|f^{-1}(y)| \geqslant 2\}$. Observe that $Y_2 = Y - Y_1$ is an open set of Y. *Case* 1: If $y \in Y_1$, then there exists an open set $P(y)$ such that

$$f^{-1}(y) \subset P(y) \subset \bigcup_{i=1}^{k} V_i,$$

$$P(y) \cap f^{-1}(p) = \varnothing \quad \text{for } p \in Y_1, \quad p \neq y.$$

Set $O = f(P(y)) \cap \bigcap_{i=1}^{k} f(V_i)$. *Case* 2: If $y \in Y_2$, then $f^{-1}(y)$ is a single point x, for which $x \in \bigcap_{i=1}^{k} V_i$. Set $O = f(\bigcap_{i=1}^{k} V_i) \cap Y_2$. In either case, O is an open set containing y such that $g(O) \subset \mathcal{V}$. Therefore g is a one-to-one map from Y into $\mathcal{C}(X)$. Hence it follows from Lemmas 1 and 2 that Y is submetrizable.

This theorem does not hold if f is an open compact map, as stated in [2, p. 55].

THEOREM 3. *Let f: $X \to Y$ be an open compact map with (*). If X has a $\overline{G_\delta(1)}$-diagonal and $\bigcup\{f^{-1}(y)$: $f^{-1}(y)$ is nontrivial$\}$ is G_δ, then Y has a G_δ-diagonal.*

PROOF. Let Y_1, Y_2 be the same as in the preceding proof. By assumption, $\bigcup \{f^{-1}(y): y \in Y_1\} = \bigcap_{n=1}^{\infty} W_n$, where W_n are open sets such that $W_{n+1} \subset W_n$ for every $n \in N$. For each $y \in Y_1$, let $P(y)$ be an open set of X such that $f^{-1}(y) \subset P(y)$ and $f^{-1}(y') \cap P(y) = \varnothing$ for every $y' \in Y_1$ with $y \neq y'$.

Since $f^{-1}(y)$ is compact and X has a $\overline{G_\delta(1)}$-diagonal, $f^{-1}(y) = \bigcap_{n=1}^{\infty} V_n(y)$, where $V_n(y)$ are open sets such that

$$V_{n+1}(y) \subset V_n(y) \subset P(y) \cap W_n \quad \text{if } y \in Y_1,$$

and

$$V_{n+1}(y) \subset V_n(y),\ V_n(y) \cap f^{-1}(Y_1) = \varnothing \quad \text{if } y \in Y_2.$$

Put for every $n \in N$ and $y \in Y$,

$$V_n'(y) = f(V_n(y)), \qquad W_n' = f(W_n).$$

Then it follows that $\{y\} = \bigcap_{n=1}^{\infty} V_n'(y)$ and $Y_1 = \bigcap_{n=1}^{\infty} W_n'$. Let $\{\mathcal{U}_n: n \in N\}$ be a sequence of open covers of X such that $\{p\} = \bigcap_{n=1}^{\infty} S(p, \mathcal{U}_n)$ for every $p \in X$. Assume that $\mathcal{U}_{n+1} < \mathcal{U}_n$ for every n. Put

$$\mathcal{V}_n = \{f(U) \cap Y_2: U \in \mathcal{U}_n\},$$
$$\mathcal{W}_n = \mathcal{V}_n \cup \{V_n'(y): y \in Y_1\}.$$

Then $\mathcal{W}_n$ is an open cover of Y. Suppose $p \neq q$, $p, q \in Y$. If $p, q \in Y_2$, then $\{p, q\} \cap W_m' = \varnothing$ for some $m \in N$. There exists an $n \in N$ with $f^{-1}(p) \notin S(f^{-1}(q), \mathcal{U}_n)$. Then we have $p \notin S(q, \mathcal{W}_k)$ for $k = \max\{m, n\}$. If $p, q \in Y_1$, then $p \notin S(q, \mathcal{W}_k)$ for every k. If $p \in Y_1$ and $q \in Y_2$, then $q \notin W_m'$ for some $m \in N$. Then we have $q \notin S(p, \mathcal{W}_m)$. Therefore in either case we have $\{p\} = \bigcap_{n=1}^{\infty} S(p, \mathcal{W}_n)$ for every $p \in Y$.

THEOREM 4. *Let $f: X \to Y$ be a perfect map with $(*)$. If X has a $\overline{G_\delta(1)}$-diagonal and $\bigcup\{f^{-1}(y): f^{-1}(y)$ is nontrivial$\}$ is G_δ, then Y has a G_δ-diagonal.*

PROOF. This is proved by the same way as in the preceding proof except that $V_n'(y)$, W_n' and $\mathcal{V}_n$ are defined as follows:

$$V_n'(y) = Y - f(X - V_n(y)), \qquad W_n' = Y - f(X - W_n),$$
$$\mathcal{V}_n = \{(Y - f(X - U)) \cap Y_2: U \in \mathcal{U}_n\}.$$

Burke's example in [1] shows that even if f is a perfect map with $(*)$ from a space X with a G_δ-diagonal onto Y, Y need not have a G_δ-diagonal. It is noted that X in his example is not developable. If we assume that X is developable, then we have Corollary 2.

COROLLARY 1. *Let $f: X \to Y$ be an open or closed map with $(*)$. If X is a perfect space (every closed set is G_δ) with a G_δ-diagonal, then Y has a G_δ-diagonal.*

PROOF. Repeat the essential part of the above proof.

COROLLARY 2. *Let $f: X \to Y$ be an open or closed map with $(*)$. If X is developable, then Y has a G_δ-diagonal.*

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