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BORNOLOGICAL SPACES OF NON-ARCHIMEDEAN VALUED FUNCTIONS WITH THE COMPACT-OPEN TOPOLOGY

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Abstract. Let F be a field with nontrivial non-Archimedean valuation of rank one and let X be a zero-dimensional Hausdorff space. The vector space $C(X, F)$ of all continuous functions from X into F is provided with the compact-open topology c. We prove that $C(X, F, c)$ is bornological if and only if X is a $\mathbf{Z}$-replete space.

Introduction. In this paper we prove the conjecture in Remark (3) of [**2**]. In [**2**] the reader may find some comments on ultraregular (zero-dimensional) spaces and $\mathbf{Z}$-replete spaces. Let β_0X be the Banaschewski compactification of X, i.e. the essentially (up to homeomorphism) unique compact ultraregular space that contains X as a dense subspace and is such that disjoint clopen sets in X have disjoint closures in β_0X. As in [**1**] v_0X is the set of all $x \in \beta_0X$ such that for each sequence $(V_n)_{n=1}^{\infty}$ of neighborhoods of x in β_0X, $\bigcap_{n=1}^{\infty} V_n \cap X \neq \varnothing$. From Theorem 9 in [**1**] we know that $v_0X = v_{\mathbf{Z}}X$.

The aim of this paper is to prove

Theorem 1. *$C(X, F, c)$ is an F-bornological space if and only if X is $\mathbf{Z}$-replete.*

The relatively easy "only if" part was obtained in [**2**, Remark 2]. So from now on we assume that X is a fixed ultraregular space with $v_0X = v_{\mathbf{Z}}X = X$ and we consider an absolutely convex subset S of $C(X, F)$ that absorbs all c-bounded sets. We want to show that S is a c-neighborhood; without loss of generality we assume $\varnothing \neq S \neq C(X, F)$.

Definition 1. If $f \in C(X, F)$, then $B(f) = \{g \in C(X, F): |g(x)| \leqslant |f(x)|$ for all $x \in X\}$.

Clearly, $B(f)$ is an absolutely convex c-bounded set.

Definition 2. $\mathcal{Q} = \{A \subset X$: A is clopen and there is a $\lambda \in F \setminus \{0\}$ with $B(\chi_A) \not\subseteq \lambda S\}$.

Here χ_A denotes the characteristic function of A; we have $\varnothing \notin \mathcal{Q}$.

Lemma 1 (slightly more general than [**2**, Lemma 1]). *If $A \notin \mathcal{Q}$ and $f \in C(X, F)$ with $f(x) = 0$ for all $x \in X \setminus A$, then $B(f) \subseteq \lambda S$ for all $\lambda \in F \setminus \{0\}$.*

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PROOF. Let $\lambda \in F \setminus \{0\}$ be given. As $B(f^2)$ is bounded, there is a $\lambda_0 \in F$ such that $|\lambda_0| \geq 1$ and $B(f^2) \subseteq \lambda_0\lambda S$. Let $W = \{x \in X: |f(x)| < |\lambda_0|\}$, a clopen set. Then $f = f \cdot \chi_W + f \cdot \chi_{A \setminus W} = f \cdot \chi_{A \cap W} + f\chi_{X \setminus W}$.

As $|f\chi_{A \cap W}| \leq |\lambda_0|\chi_A$ and $A \notin \mathcal{Q}$, $B(f\chi_{A \cap W}) \subseteq \lambda_0 B(\chi_A) \subseteq \lambda_0\lambda_0^{-1}\lambda S = \lambda S$. Hence as $|f\chi_{X \setminus W}| \leq |\lambda_0^{-1}f^2|$, $B(f) \subseteq B(f\chi_{A \cap W}) + B(f\chi_{X \setminus W}) \subseteq \lambda S + \lambda_0^{-1}\lambda_0 S \subseteq \lambda S$.

LEMMA 2. *If $A \in \mathcal{Q}$, $A = \bigcup_{i=1}^{\infty} A_i$ with all A_i clopen, then there is an i with $A_i \in \mathcal{Q}$.*

PROOF. Cf. [2, Lemma 2].

LEMMA 3. *If $(A_i)_{i=1}^{\infty}$ is a family of disjoint clopen sets, and if $\bigcup_{i=1}^{\infty} A_i$ is clopen, then at most a finite number of A_i are in $\mathcal{Q}$.*

PROOF. Cf. [2, Lemma 3]. Remark, however, that our case requires $\bigcup_{i=1}^{\infty} A_i$ to be a clopen set.

PROOF OF THE "IF" PART OF THEOREM 1. Let $\mathcal{K}$ be the family of all clopen subsets V of X such that $X \setminus V \notin \mathcal{Q}$. Then $\mathcal{K} \subseteq \mathcal{Q}$ by Lemma 2. By Lemma 1, $X \in \mathcal{K}$ and $\phi \notin \mathcal{K}$. Clearly, any clopen subset of X containing a member of $\mathcal{K}$ belongs to $\mathcal{K}$. Finally, if $V, W \in \mathcal{K}$, then $V \cap W \in \mathcal{K}$, for as $X \setminus (V \cap W) = (X \setminus V) \cup (X \setminus W)$, $X \setminus (V \cap W) \notin \mathcal{Q}$ by Lemma 2.

Thus $\mathcal{K}$ is a filter base of clopen sets. Let K be the adherence of $\mathcal{K}$ in $\beta_0 X$. Thus K is a nonempty, compact subset of $\beta_0 X$. We shall show that $K \subseteq v_0 X = X$. Assume, on the contrary, that $a \in K \setminus v_0 X$. Then there is a decreasing sequence $(V_n)_{n \geq 0}$ of clopen neighborhoods of a in $\beta_0 X$ such that $V_0 = \beta_0 X$ and $(\bigcap_{n=0}^{\infty} V_n) \cap X = \varnothing$. Hence $X = \bigcup_{n=1}^{\infty}((V_n \setminus V_{n+1}) \cap X)$.

By Lemma 3, there exists $m \geq 0$ such that $(V_n \setminus V_{n+1}) \cap X \notin \mathcal{Q}$ for all $n \geq m$. Consequently, as

$$V_m \cap X = \bigcup_{n=m}^{\infty} ((V_n \setminus V_{n+1}) \cap X), \qquad V_m \cap X \notin \mathcal{Q},$$

by Lemma 2, whence $X \setminus V_m \in \mathcal{K}$. Thus $a \in \overline{X \setminus V_m} \cap \overline{X \cap V_m}$ (closures in $\beta_0 X$). However, $X \setminus V_m$ and $X \cap V_m$ are disjoint clopen sets in X so that their closures in $\beta_0 X$ are disjoint. This contradiction establishes that $K \subseteq v_0 X = X$.

Now clearly $K = \bigcap \mathcal{K}$. As $B(\chi_X)$ is bounded, there is a $\lambda \neq 0$ such that $B(\chi_X) \subseteq \lambda S$. As K is compact, we need only show that if $|f(x)| < |\lambda|^{-1}$ for all $x \in K$, then $f \in S$. Let $W = \{x \in X: |f(x)| \leq |\lambda|^{-1}\}$, a clopen subset of X. The closure $\overline{W}$ of W in $\beta_0 X$ is a neighborhood of K. Since $(\bigcap \mathcal{K}) \cap (\beta_0 X \setminus \overline{W}) = \varnothing$ and $\beta_0 X \setminus \overline{W}$ is compact, there is a $V \in \mathcal{K}$ with $V \subset \overline{W}$. Hence $V \subseteq W$ and $X \setminus W \notin \mathcal{Q}$. By Lemma 1 applied to $X \setminus W$, $B(f \cdot \chi_{X \setminus W}) \subseteq S$, and $B(f \cdot \chi_W) \subseteq B(\lambda^{-1}\chi_X) \subseteq S$, so

$$B(f) \subseteq B(f \cdot \chi_{X \setminus W}) + B(f \cdot \chi_W) \subseteq S + S \subseteq S.$$

In particular, $f \in S$.

Final Remarks. This paper was greatly improved and simplified thanks to the comments of Professor S. Warner. He also remarked that the principal result of [2] can be established in this way. Indeed, suppose that, in addition to the assumptions made on S in this paper, S absorbs all pointwisely bounded sets. It suffices to prove that K is finite. By Lemma 4 of [2], $X = A_1 \cup A_2 \cup \cdots \cup A_n$ where each $A_i \in \mathcal{Q}$ and A_i is not the disjoint union of two members of $\mathcal{Q}$. We shall prove that $K \cap A_i$ has at most one point. Suppose $a, b \in K \cap A_i$, $a \neq b$. Let V be clopen such that $a \in V$, $b \in X \setminus V$. Either $A_i \cap V \notin \mathcal{Q}$ or $A_i \setminus V \notin \mathcal{Q}$. We may assume $A_i \cap V \notin \mathcal{Q}$. Then $X \setminus (A_i \cap V) \in \mathcal{K}$ so that $a \in K \subseteq X \setminus (A_i \cap V)$, which is a contradiction.

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