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A PROOF OF THE PRINCIPLE OF LOCAL REFLEXIVITY

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ABSTRACT. A quite elementary proof of the principle of local reflexivity is given.

Our purpose here is to give a proof of the "principle of local reflexivity" (using only the various forms of the Hahn-Banach theorem) as given in [1] and in an improved version in [2]. Our notation is standard. By X, Y and Z we shall denote Banach spaces and J_X will denote the canonical embedding of X into its second dual X''. An operator is a continuous linear function.

We shall require only the three following lemmas.

LEMMA 1. *Let $T: X \to Y$ be a closed operator. If x'' is in X'' and y is in Y such that $T''x'' = J_Y y$ then, for any $\partial > 0$ there exists an x in X such that*

$$\|x\| < (1 + \partial)\|x''\|$$

and $Tx = y$.

LEMMA 2. *Let $T: X \to Y$ and $S: X \to Z$ be operators such that T is closed and S has finite rank. Then $U: X \to Y \times Z$ defined by $Ux = (Tx, Sx)$ is a closed operator.*

LEMMA 3. *Let $0 < \partial < \frac{1}{4}$ and $T: X \to Y$ be an operator such that X is finite dimensional and*

$$(1 + \partial)^{-1} \leqslant \|Tx_i\| \leqslant (1 + \partial)$$

where $\{x_i\}$ is any ∂-net for the unit sphere of X. Then T is invertible and

$$\|T\| \, \|T^{-1}\| \leqslant \left(\frac{1 + \partial}{1 - \partial}\right)\left(\frac{1}{1 + \partial} - \frac{\partial(1 + \partial)}{1 - \partial}\right)^{-1} = \vartheta(\partial).$$

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Lemma 1 follows immediately from the separation theorem. Lemma 2 is easily proved by observing that U' is closed and Lemma 3 is a routine computation using the triangle inequality.

Theorem [1], [2]. *Let E and F be finite dimensional subspaces of X'' and X', respectively, and let $\varepsilon > 0$. Then there exist an operator T: $E \to X$ such that $\|T\|\,\|T^{-1}\| < 1 + \varepsilon$, $x'(Tx'') = x''(x')$ for all x'' in E and all x' in F, and $Tx'' = x$ if $J_X x = x''$ is in E.*

Proof. Choose $\partial > 0$ so that $\vartheta(\partial) < 1 + \varepsilon$ where ϑ is as in Lemma 3. Choose norm one elements $a_1', a_2', \ldots, a_m'$ in X' containing a basis of F and such that

$$\|x''\| < (1 + \partial) \sup_j \left|x''(a_j')\right|$$

for all x'' in E. Choose $b_1'', b_2'', \ldots, b_n''$ a ∂-net for the unit sphere of E such that $b_1'', \ldots, b_k''$ is a basis for $J_X X \cap E$ and $b_1'', \ldots, b_r''$, $r \geqslant k$, is a basis for E. Then, for $1 \leqslant p \leqslant q = n - r$, we have the unique scalars $\{t_{p,i}\}$, $1 \leqslant i \leqslant r$, such that

$$b_{r+p}'' = \sum_{1 \leqslant i \leqslant r} t_{p,i} b_i''.$$

Define for $1 \leqslant p \leqslant q$

$$s_{p,i} = \begin{cases} t_{p,i}, & i \leqslant r, \\ -1, & i = r + p, \\ 0, & r < i \leqslant n \text{ and } i \neq r + p. \end{cases}$$

Define A_0: $X^n \to X^{k+q}$ by

$$A_0(x_1, \ldots, x_n) = \Big(x_1, \ldots, x_k; \Big(\sum_{1 \leqslant i \leqslant n} s_{p,i} x_i\Big)\Big)$$

for $1 \leqslant p \leqslant q$ where X^n and X^{k+q} are the usual product spaces with the sup norm. The operator A_0 is onto since the matrix $(s_{p,i})$ has rank q. Define A: $X^n \to Z = X^{k+q} \times \mathbf{C}^{nm}$ by

$$A(x_1, \ldots, x_n) = \big(A_0(x_1, \ldots, x_n); \big(a_j'(x_i)\big)\big)$$

for $1 \leqslant j \leqslant m$ and $1 \leqslant i \leqslant n$. By Lemma 2, A is a closed operator. Observe that $A''(b_1'', \ldots, b_n'')$ is in $J_Z Z$. Therefore, by Lemma 1, there exists $(b_1, \ldots, b_n)$ in X^n,

$$\sup_i \|b_i\| < (1 + \partial) \sup_i \|b_i''\| = 1 + \partial,$$

such that $J_Z A(b_1, \ldots, b_n) = A''(b_1'', \ldots, b_n'')$. Define the operator T: $E \to X$ such that $Tb_i'' = b_i$ for $1 \leqslant i \leqslant r$. For $1 \leqslant p \leqslant q$, we have that $\Sigma_{1 \leqslant i \leqslant n} s_{p,i} b_i'' = 0$ and $\Sigma_{1 \leqslant i \leqslant n} s_{p,i} b_i = 0$ which gives that $Tb_i'' = b_i$ also for $r < i \leqslant n$. To apply Lemma 3 and complete the proof we need only observe that for each i,

$$\|Tb_i''\| \geqslant \sup_j |a_j'(Tb_i'')| = \sup_j |b_i''(a_j')| \geqslant (1 + \partial)^{-1}.$$

This proof was presented at the Functional Analysis Conference at Oberwolfach in October, 1974.

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