PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 78, Number 2, February 1980

HAUSDORFF MATRICES AS BOUNDED OPERATORS OVER l

B. E. RHOADES

ABSTRACT. A necessary and sufficient condition is obtained for an arbitrary Hausdorff matrix to belong to $B(l)$. It is then shown that every conservative quasi-Hausdorff matrix is of type M.

Let (H, μ) denote the Hausdorff method with generating sequence $\mu = \{\mu_n\}$, $l = \{\{x_n\} \mid \Sigma_n |x_n| < \infty\}$, $B(l)$ the algebra of bounded linear operators on l. A necessary and sufficient condition is obtained for an arbitrary Hausdorff method to belong to $B(l)$. It is then shown that every conservative quasi-Hausdorff matrix is of type M.

An infinite matrix is called triangular if it has only zeros above the main diagonal. Let $B(c)$ denote the algebra of bounded linear operators in c, the space of convergent sequences.

LEMMA. *Let A be a triangular matrix satisfying:*
(1) $A \in B(l)$,
(2) $t_n^* = \Sigma_{k=0}^n |a_{nk}|$ *is monotone increasing in* n,
(3) $\lim_n t_n$ *exists, where* $t_n = \Sigma_{k=0}^n a_{nk}$.
Then $A \in B(c)$.

PROOF. Condition (1) is equivalent to $\sup_k \Sigma_{n=k}^\infty |a_{nk}| < \infty$, which implies $\Sigma_{n=k}^N |a_{nk}| \leqslant M$ for every $N > k$, where M is independent of N and k. Summing k over $[0, N]$ yields $\Sigma_{k=0}^N \Sigma_{n=k}^N |a_{nk}| \leqslant M(N+1)$. Interchanging the order of summation gives

$$\sum_{n=0}^{N} t_n^* / (N+1) \leqslant M. \tag{4}$$

The left-hand side of (4) is the Nth term of the Cesàro transform of order one, $(C, 1)$, of the sequence $\{t_n^*\}$. Since $\{t_n^*\}$ is monotone increasing, the norm of A in $B(c)$ is $\|A\|_c = \lim_n t_n^*$. If $\|A\|_c = \infty$, then the total regularity of $(C, 1)$ implies that the l.h.s. of (4) tends to ∞ as $N \to \infty$, a contradiction. Therefore $\|A\|_c$ is finite. Condition (1) implies A has zero column limits and (3) assures the existence of the limit of the row sums. Therefore $A \in B(c)$.

Received by the editors March 21, 1979.
AMS (MOS) subject classifications (1970). Primary 40G05; Secondary 40C05.
Key words and phrases. Hausdorff means, quasi-Hausdorff means, type M.

THEOREM 1. *Let H be a Hausdorff matrix. Then $H \in B(l)$ if and only if μ is a moment sequence satisfying*

$$\int_0^1 |d\beta(t)|/t < \infty. \tag{5}$$

PROOF. In [3, p. 279] it is shown that, if μ is a moment sequence, then $H \in B(l)$ if and only if μ satisfies (5). The theorem in Hardy actually treats conservative quasi-Hausdorff methods (H^*, μ), but (H^*, μ) is merely the matrix transpose of (H, μ), so that the norm condition for the regularity of H^* is the same as the norm condition for $H \in B(l)$.

It remains to show that, if $H \in B(l)$, then μ is a moment sequence.

From [3, p. 254] or [4, Lemma 2, p. 177], $\{t_n^*\}$ is monotone increasing in n. Each row sum of H adds up to μ_0, so (3) of the Lemma is satisfied. Applying the Lemma, $H \in B(c)$, which by [3, p. 260] is equivalent to μ being a moment sequence.

Let $A \in B(c)$, A a matrix. Then A is said to be of type M if the only solution of $tA = 0$, $t \in l$, is $t = 0$.

THEOREM 2. *Let $(H^*, \mu) \in B(c)$. Then (H^*, μ) is of type M.*

In [5] it was shown that H^* is of type M, provided at most a finite number of the μ_n are zero. We now provide a different proof, which removes that restriction.

Since H^* is the matrix transpose of H, the condition $H^* \in B(c)$ is equivalent to $H \in B(l)$. By Theorem 1, μ is a moment sequence. Thus there exists a function $\beta(t) \in BV[0, 1]$ such that $\mu_n = \int_0^1 t^n \, d\beta(t)$. Define

$$F(z) = \int_0^1 t^z \, d\beta(t).$$

Then F is analytic in $\operatorname{Re} z > 0$ and continuous on $\operatorname{Im} z = 0$. Moreover, if b_i denote the real zeros of F in $\operatorname{Re} z > 0$, they satisfy $\Sigma\, 1/b_i < \infty$, by a result of Carleman [1]. To say that H^* is of type M is equivalent to saying that H is 1-1 on l.

Since $H = \delta\mu\delta$, where δ is the triangular matrix of signed binomial coefficients and μ is a diagonal matrix with diagonal entries μ_n, we may use associativity of multiplication and the fact that δ is its own inverse, to obtain $\mu\delta t = 0$; i.e., $\mu_n \Delta^n t_0 = 0$ for $n = 0, 1, 2, \ldots$. Now appeal to [2, Theorem 1] to conclude that t is a constant sequence. The result follows, since the only constant sequence in l is the zero sequence.

Theorems 1 and 2 are also true for generalized Hausdorff matrices of the form

$$h_{nk} = \binom{n+\alpha}{n-k} \Delta^{n-k} \mu_k, \quad \text{with } \alpha \geqslant 0.$$

References

1. T. Carleman, *Über die Approximation analytischer Funktionen durch Aggregate vorgegebener Potenzen*, Ark. Mat. **17** (1922), no. 9.

2. W. H. J. Fuchs, *A theorem on finite differences with an application to the theory of Hausdorff summability*, Proc. Cambridge Philos. Soc. **40** (1944), 188–196.

3. G. H. Hardy, *Divergent series*, Oxford Univ. Press, London, 1949.

4. A. Jakimovski, B. E. Rhoades and J. Tzimbalario, *Hausdorff matrices as bounded operators over l^p*, Math. Z. **138** (1974), 173–181.

5. B. E. Rhoades, *Type M for quasi-Hausdorff matrices*, Proc. Cambridge Philos. Soc. **68** (1970), 601–604.

Department of Mathematics, Indiana University, Bloomington, Indiana 47405