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THE CLOSED SOCLE OF AN AZUMAYA ALGEBRA

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ABSTRACT. If R is a Noetherian ring and A is an Azumaya algebra over R then an ideal $H(A)$ in R, called the closed socle of A, is defined and it is shown that $H(A)$ is independent of the representative A in the Brauer group of R. When R is a domain, the behavior of $H(A)$ under localization and passage to the quotient field is studied, and $H(A)$ is calculated when R is the affine ring of a real curve.

Let R denote a Noetherian, integrally closed domain and A an Azumaya (central separable) algebra over R. In [**6**], D. Haile associated to A an ideal in R which he called the closed socle of A and which we denote $H(A)$. In this note we show how to define the closed socle of an Azumaya algebra over any commutative Noetherian ring, give simplified proofs of results slightly more general than those in [**6**], and calculate the closed socle of an Azumaya algebra over the affine ring of a real curve. More specifically, if R is a commutative Noetherian ring and A is an Azumaya algebra over R then $H(A)$ is defined and is independent of the choice of representative A in its class in the Brauer group $B(R)$ of R. If R is a local Noetherian domain with quotient field F and maximal ideal m and $\Sigma = A \otimes F$, then Index $(\Sigma) \geqslant \text{Index}(A/mA)$ when $H(A) = R$. Thus, if $\Sigma = M_n(F)$ then $H(A) = R$ if and only if $A = M_n(R)$. Also, if A/mA is a division algebra then A is a maximal order in a division algebra over F when $H(A) = R$. A localization result is proved and consequences of these results for Noetherian domains are derived. If R is the affine ring of a real curve X and A is an Azumaya algebra over R then $H(A) \subseteq \Pi P_x$ where P_x is the prime ideal in R corresponding to a point $x \in X$ and x runs over the singular points $x \in X$ which are isolated in the strong topology and for which A/P_xA is not in the trivial class of $B(R/P_x)$. Throughout all unexplained terminology and notation is as in [**4**], and $\otimes$ always means $\otimes_R$.

1. We begin by extending the definition of the closed socle given in [**6**]. Let R be a domain with quotient field F, let A be an Azumaya algebra over R, and let $\Sigma = A \otimes F = A \cdot F$. A left ideal L in A is pure over R in case $ra \in L$ for $0 \neq r \in R$ and $a \in A$ implies $a \in L$ ([**2**, p.199]). It is easy to show that there is a one-to-one order preserving correspondence between the R-pure left ideals L in A and the left ideals L^1 in Σ by $L \to L \cdot F$ and $L^1 \to L^1 \cap A$. It follows that minimal pure left ideals exist in A. Let I be the sum of the minimal R-pure left ideals in A, then the closed socle of A is defined to be $I \cap R$ and is denoted $H(A)$. Observe that the set I in A is actually a two-sided ideal in A. It is clear that I is a left ideal in A, and for $a \in A$ and minimal pure left ideal L of A either $La = 0$ or

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$(La)F = (LF)a$ is a minimal left ideal of Σ, since $La \subseteq (LF)a \cap A$ and $(LF)a \cap A$ is a two-sided ideal in A. By the one-to-one correspondence between two-sided ideals of A and R ([**4**, 3.7, p. 54]) it follows that $I = A \cdot H(A)$. This corresponds to the construction of the closed socle for normal domains given in [**6**]. Now assume R is a reduced Noetherian ring. Then $(0) = \bigcap_{i=1}^{n} P_i$, where the P_i are a uniquely determined set of prime ideals so that the intersection is irredundant. If A is an Azumaya algebra over R let $A_i = R/P_i \otimes A$, then A_i is an Azumaya algebra over the domain R/P_i. From the above $H(A_i)$ is an ideal in R/P_i whose natural inverse image is an ideal I_i in R. We define the closed socle $H(A)$ of A to be $\bigcap_{i=1}^{n} I_i$. If R is just a commutative Noetherian ring let N be the nil radical of R, and let A be an Azumaya algebra over R. Above we defined $H(A/NA)$ in R/N. We define the closed socle $H(A)$ of A to be the natural inverse image of $H(A/NA)$ in R. The calculation of the closed socle of an Azumaya algebra over the affine ring of a real curve given later illustrates the naturalness of this definition.

LEMMA 1. *Let R be a domain and P a prime ideal in R. Let A be an Azumaya algebra over R, then $H(R_P \otimes A) = R_P \otimes H(A)$ where R_P denotes the localization of R at P.*

PROOF. One can check that there is a one-to-one correspondence between the minimal pure left ideals L of A and the minimal pure left ideals L^1 of $R_P \otimes A = AR_P$ by $L \to LR_P$ and $L^1 \to A \cap L^1$. Let I be the sum of the minimal pure left ideals of A and I^1 the sum of the minimal pure left ideals in A^1. Then $I \cdot R_P = \Sigma LR_P$ where L runs through the minimal pure left ideals in A. Thus $IR_P = I^1$. Now one can check that $H(A)R_P = H(R_P \otimes A)$.

We note here that if S is a commutative R-algebra and A is an Azumaya R-algebra it may not be the case that $H(S \otimes A) = S \otimes H(A)$. For example let $\mathbf{R}$ denote the field of real numbers, let $R = \mathbf{R}[x,y]/(x^2+y^2)$ and let S be the integral closure of R. We later show that if A is the Azumaya algebra generated by elements i, j subject to $i^2 = j^2 = -1$ and $ij = -ji$ and P is the maximal ideal in R generated by $\{x, y\}$ then $H(A) \subseteq P$; but $H(S \otimes A) = S$ so $S \otimes H(A) \neq H(S \otimes A)$.

LEMMA 2. *Let A and A^1 be two Azumaya algebras in the same class of the Brauer group of the Noetherian ring R, then $H(A) = H(A^1)$.*

PROOF. Assume R is a local domain. Then there is an Azumaya R-algebra D with no idempotents other than 0 and 1 and positive integers n and k with $A \cong M_n(D)$, $A^1 = M_k(D)$ by Corollary 1 of [**3**]. In this case it suffices to show $H(D) = H(M_n(D))$. Let F be the quotient field of R and $\Sigma = F \otimes D = F \cdot D$. Then $F \otimes M_n(D) = M_n(\Sigma)$. Let e_{ij} be the matrix in $M_n(\Sigma)$ with a 1 in the i,j entry and 0 elsewhere. Let L be a minimal pure left ideal in D so $L^1 = \Sigma L$ is a minimal left ideal in Σ. Then $M_n(L) = \bigoplus_{i=1}^{n} M_n(L)e_{ii}$ and $M_n(\Sigma) \cdot M_n(L) = M_n(\Sigma \cdot L) = \bigoplus_{i=1}^{n} M_n(\Sigma L)e_{ii}$. Moreover, since L is R-pure in Σ, $M_n(L)e_{ii}$ is R-pure in $M_n(\Sigma)$ so $M_n(\Sigma L)e_{ii} \cap M_n(D) = M_n(L)e_{ii}$. Finally, $M_n(\Sigma L)e_{ii}$ is a minimal left ideal in $M_n(\Sigma)$ since $\Sigma \cdot L$ is a minimal left ideal in Σ. Thus

$$M_n(L) = M_n(\Sigma L) \cap M_n(D) = \left(\bigoplus_{i=1}^{n} M_n(\Sigma L)e_{ii} \right) \cap M_n(D)$$

$$\supseteq \bigoplus_{i=1}^{n} M_n(\Sigma L)e_{ii} \cap M_n(D) = \bigoplus_{i=1}^{n} M_n(L)e_{ii} = M_n(L).$$

Thus $M_n(L)$ is contained in the sum of the minimal pure left ideals in $M_n(D)$ so $H(D) \subseteq H(M_n(D))$. For the reverse inclusion let I be the sum of the minimal pure left ideals L of D. Let J' be the sum of minimal pure left ideals in $M_n(D)$. By 3.5, p. 22 in [4] we know $J' = M_n(J)$ for some two-sided ideal J in D. We have already shown $I \subseteq J$. Let (x_{ij}) be an $n \times n$ matrix in $M_n(D)$ with $(x_{ij}) \notin M_n(I)$ yet $(x_{ij}) \in L^1$ for some minimal pure left ideal L^1 of $M_n(D)$. If $x_{ij} \notin I$ then $(e_{ii})(x_{ij})(e_{jj})$ is contained in $M_n(D)$ and in the minimal left ideal L^1e_{jj}. Thus if X_{ij} is the matrix whose ijth entry is x_{ij} and all others are 0 then $X_{ij} \in L^1e_{jj} \cap M_n(D)$ is in a minimal pure left ideal in $M_n(D)$, so $M_n(\Sigma)X_{ij}$ is a minimal left ideal in $M_n(\Sigma)$. Thus Σx_{ij} must be a minimal left ideal in Σ. Thus $x_{ij} \in \Sigma x_{ij} \cap D \subseteq I$ which contradicts the choice of x_{ij}.

Now let R be any commutative Noetherian ring with nil radical N. If A is equivalent to A^1 in $B(R)$ then A/NA is equivalent to A^1/N^1 in $B(R/N)$ so it suffices to show $H(A/N(A)) = H(A^1/NA^1)$. Similarly, we can assume R is a domain for if we write (0) as the irredundant intersection of a unique set $\{P_i\}_{i=1}^n$ of prime ideals and if $H(A/P_iA) = H(A^1/P_iA^1)$ for all i then $H(A) = H(A^1)$. If R is a Noetherian domain and P is any prime ideal in R we know $R_P \otimes H(A) = R_P \otimes H(A^1)$ by the first part of the proof since $R_P \otimes A$ is equivalent to $R_P \otimes A^1$ in $B(R_P)$. Thus if $H(A) = \bigcap_{i=1}^n Q_i^{n_i}$ and $H(A^1) = \bigcap_{i=1}^n Q_i^{m_i}$ where $\{Q_i\}$ is an irredundant collection of prime ideals and m_i, $n_i \geqslant 0$ then $m_i = n_i$ for all i so we always have $H(A) = H(A^1)$.

If A is an Azumaya algebra over a field F then the index of A is the square root of the dimension of the division algebra part of A over F. The next result is a generalization (with an easier proof) of Theorem 4.6 of [6].

THEOREM 1. *Let R be a local Noetherian domain with maximal ideal m and quotient field F. Let A be an Azumaya algebra over R and $\Sigma = F \otimes A$. If $H(A) = R$ then* Index $\Sigma \geqslant$ Index A/mA. *Moreover, if* Index $\Sigma =$ Index A/mA *then $A = M_n(B)$ where B is an Azumaya R-algebra such that $F \otimes B$ is a division algebra.*

PROOF. Assume $H(A) = R$. Then there is a minimal pure left ideal L of A with $L \not\subseteq mA$. Let $A_0 = \Sigma La$ where the sum runs over those $a \in A$ such that $La \not\subseteq mA$. Then A_0 is an R-submodule of A and $(A_0 + mA)/mA$ is a two-sided ideal in A/mA. Since A/mA is simple, $(A_0 + mA)/mA = A/mA$, so by Nakayama's lemma, $A_0 = A$. Therefore $A = \Sigma La$ where the sum is taken over finitely many $a \in A$ such that $La \not\subseteq mA$. Let $\overline{La} = F \cdot La$. Then $\Sigma = \overline{La_1} \oplus \cdots \oplus \overline{La_n}$. Thus $A \supseteq La_1 \oplus \cdots \oplus La_n = A_1$ and each $La_i \not\subseteq mA_i$ and A_1 is an R-submodule of A. Note that $(A_1 + mA)/mA = (La_1 + mA)/mA \oplus \cdots \oplus (La_n + mA)/mA$ and each summand on the right is a nonzero ideal in A/mA. This proves Index $\Sigma \geqslant$ Index A/mA. If Index $\Sigma =$ Index A/mA then $(A_1 + mA)/mA = A/mA$ so by

Nakayama's lemma $A_1 = A$. Let $B^0 = \mathrm{Hom}_A(La_1, La_1)$. Then B is an Azumaya algebra over R and $A = \mathrm{Hom}_{B^0}(La_1, La_1) = M_n(B)$, with B a maximal order in the division algebra component of Σ.

COROLLARY 1. *Let R be a local Noetherian domain with quotient field F, and let A be an Azumaya R-algebra. If $F \otimes A \cong M_n(F)$ then $A \cong M_n(R)$ if and only if $H(A) = R$.*

PROOF. If $A \cong M_n(R)$ then it is easy to see that $H(A) = R$ (this also follows from Lemma 2). Conversely, Index $F \otimes A = 1 \geqslant$ Index A/mA so the inequality is an equality and the result follows from Theorem 1.

COROLLARY 2. *Let R be a Noetherian domain with field of quotients F. Suppose A is an Azumaya R-algebra with $F \otimes A \cong M_n(F)$. Let P be a prime ideal in R, then $R_P \otimes A \cong M_n(R_P)$ if and only if $H(A) \not\subseteq P$.*

PROOF. Combine Lemma 1 and Corollary 1.

COROLLARY 3. *Let R and A be as in Theorem 1. If A/mA is a division algebra and $H(A) = R$ then $F \otimes A$ is a division algebra.*

PROOF. By Theorem 1, Index $F \otimes A \geqslant$ Index A/mA since $[\text{Index } A/mA]^2 = \mathrm{Rank}_{R/m}(A/mA) = \mathrm{Rank}_R(A) = \mathrm{Rank}_F(F \otimes A) \geqslant [\text{Index } F \otimes A]^2$. Thus $[\text{Index } F \otimes A]^2 = \mathrm{Rank}_F(A)$ and $F \otimes A$ is a division algebra.

THEOREM 2. *Let R be the affine ring of a real curve X. For each closed point $x \in X$ let P_x be the corresponding maximal ideal in R. Let A be an Azumaya R-algebra. Then the longest product of ideals P_x containing $H(A)$ has factors P_x satisfying all of the following.*

1. *x is a real singular point on X.*
2. *x is isolated in the strong topology on the real points of the irreducible component X_i of X containing x.*
3. *A/P_xA does not represent the trivial class in $B(R/P_x)$.*

PROOF. For each $x \in X$ let $R(x) = R/P_x$ and $A(x) = A/P_xA$. Since X is a real curve $R(x)$ is either the real or complex numbers. If $R(x)$ is the real number $\mathbf{R}$ then either $A(x)$ is a matrix algebra over $\mathbf{R}$ or $A(x)$ is a matrix algebra over the division algebra of real quaternions. In [5] the Azumaya algebras A over R are characterized as continuously parameterized systems of Azumaya algebras $A(x)$ over the real points $x \in X$ with the strong topology. If X is irreducible then R is a domain and the quotient field F of R is the field of rational functions on X. By Remark 3.4 of [5] the real points x for which $A(x)$ is a matrix algebra over the quaternions, yet $F \otimes_{R(x)} A(x)$ is a matrix algebra over F, are precisely the real singular points $x \in X$ which are isolated in the strong topology and for which $A(x)$ is a matrix algebra over the quaternions. Combining Theorem 1 and Lemma 1 we see that $A(x)$ is a matrix algebra over the quaternions yet $F \otimes A(x)$ is a matrix algebra over F if and only if P_x contains $H(A)$. If X is reducible we can assume R is reduced. Let $(0) = \bigcap_{i=1}^{n} P_i$ where $\{P_i\}$ is an irredundant set of prime ideals of R. The rings R/P_i are the affine rings of the irreducible components X_i of X. If

$A_i = R/P_i \otimes A$ then $H(A_i)$ is contained in the product of the maximal ideals which correspond to the real singular points x_i on X_i which are isolated in the strong topology on X_i and for which $A(x)$ is a matrix algebra over the quaternion algebra. This completes the proof of the theorem.

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