PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 78, Number 3, March 1980

A LEMMA ON EXTENSIONS OF ABELIAN GROUPS

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ABSTRACT. We prove: If H is a Fuchs 5 group, then for all groups G containing H it follows that $H = H_1 \oplus H_2$, $G = H_1 \oplus G_2$ such that $H_2 \subset G_2$ and $|G_2| \leqslant |G/H| \cdot \aleph_0$. There are a variety of applications.

1. Introduction. In this paper, all groups are abelian. If a group H is extended by a group K and $|K|$ is small, then K should be attached to a small part of H only. This feeling is substantiated in the Useful Lemma below provided H is "Fuchs 5".

DEFINITION [2, p. 446]. A group H is a *Fuchs* 5 *group* if every subgroup L of H can be imbedded in a direct summand of H of cardinality $\leqslant |L| \cdot \aleph_0$.

Direct sums of countable groups are Fuchs 5 groups. It seems to be unknown whether Fuchs 5 groups are necessarily direct sums of countable groups.

In the following lemma, K should be thought of as the group generated by a set of representatives for G/H.

USEFUL LEMMA. *Let H and K be subgroups of G such that H is a Fuchs* 5 *group and $H + K = G$. Then $H = H_1 \oplus H_2$ and $G = H_1 \oplus G_2$ such that $H \cap K \subset H_2 \subset G_2$ and $|G_2| \leqslant |K| \cdot \aleph_0$.*

This lemma–which has a short and elementary verification–provides instant proofs of several known theorems and possibly of some new ones. Furthermore, it characterizes Fuchs 5 groups.

PROPOSITION. *Let H be a group such that for every group $G \supset H$ and every subgroup K of G with $H + K = G$ the conclusions of the Useful Lemma are valid. Then H is a Fuchs* 5 *group.*

I presented the Useful Lemma in a seminar at New Mexico State University and received many helpful comments from the audience. In particular, Professors Richman and Walker came up with the elementary proof given below. My original proof used some basic homological algebra.

2. Proofs and applications. We first prove the lemma. Since H is Fuchs 5, $H = H_1 \oplus H_2$ with $H \cap K \subset H_2$ and $|H_2| \leqslant |H \cap K| \cdot \aleph_0 \leqslant |K| \cdot \aleph_0$. Let $G_2 = H_2 + K$. Then clearly

$$|G_2| \leqslant |K| \cdot \aleph_0 \quad \text{and} \quad H_1 + G_2 = H_1 + H_2 + K = H + K = G.$$

Received by the editors March 7, 1979.
AMS (MOS) subject classifications (1970). Primary 20K35.
Key words and phrases. Fuchs 5 group, extension, summand.

Suppose $x \in H_1 \cap G_2$. Then $x = h + k$, $h \in H_2$, $k \in K$. Hence $x - h = k \in H \cap K \subset H_2$, and $x = h + k \in H_1 \cap H_2 = 0$. So $x = 0$ and $G = H_1 \oplus G_2$. □

As a first application we prove a well-known theorem.

2.1. PROPOSITION [1, 68.3, Vol. II, p. 16]. *If G is a separable p-group and B is a basic subgroup of G such that G/B is countable, then G is a direct sum of cyclic groups.*

PROOF. Let K be the subgroup generated by a set of representatives of G/B. Then $|K| \leq \aleph_0$. Also B is Fuchs 5. By the Useful Lemma, $G = B_1 \oplus G_2$ where $B_1 \subset B$ and G_2 is countable. Hence B_1 is a direct sum of cyclic groups and so is G_2 as a countable p-group with no elements of infinite height. □

The next proposition is similar to and implies Proposition 2.1.

2.2. PROPOSITION. *If G is any group containing a direct sum of countable groups B as a subgroup of countable index, then G is a direct sum of countable groups.*

PROOF. By the Useful Lemma, $G = B_1 \oplus G_2$ with G_2 countable and B_1 a direct summand of B. By the Kaplanski-Walker theorem [1, 9.10, Vol. I, p. 49], B_1 is a direct sum of countable groups. □

The next proposition generalizes the basic Theorem 1 in [3] from p-groups to arbitrary groups.

2.3. PROPOSITION. *Let G be any group with p-basic subgroup B. Then $B = B_1 \oplus B_2$, $G = B_1 \oplus G_2$ such that B_2 is a p-basic subgroup of G_2, $|G_2| \leq |G/B| \cdot \aleph_0$ and $G/B \cong G_2/B_2$.*

PROOF. Obvious from the Useful Lemma.

The last result is particularly interesting when $|G/B| < |G|$. It then implies that G has a summand of cardinality $|G|$ which is a direct sum of cyclic groups.

There is an interesting application to torsion-free groups which may be new.

2.4. PROPOSITION. *If G is separable torsion-free group and H a completely decomposable subgroup of countable index, then G is completely decomposable.*

PROOF. By the lemma, $G = H_1 \oplus G_2$ where G_2 is countable and H_1 is a direct summand of H. Now H_1 is completely decomposable as a direct summand of such a group [1, 86.7, Vol. II, p. 114], G_2 is separable as a direct summand of such a group [1, 87.5, Vol. II, p. 120] and finally, countable separable torsion-free groups are completely decomposable [1, 87.1, Vol. II, p. 118], so G_2 is completely decomposable. □

2.5. PROPOSITION. *If G is indecomposable and H is a proper Fuchs 5 subgroup of G, then $|G/H| \cdot \aleph_0 = |G|$.*

PROOF. $G = H_1 \oplus G_2$ with $H_1 \subset H$ and $|G_2| \leq |G/H| \cdot \aleph_0$. Since H_1 is a proper subgroup and G is indecomposable, it follows that $G = G_2$. So $|G| \leq |G/H| \cdot \aleph_0 \leq |G|$. □

We include two more applications.

2.6. PROPOSITION. *If G is any group such that $p^\alpha G$ is Fuchs 5 for some $\alpha \geq 1$ and $\aleph_0 \leq |G/p^\alpha G|$, then $|G| = |G/p^\alpha G|$ or G has a p-divisible direct summand of cardinality $|G|$.*

PROOF. $G = G_1 \oplus G_2$, $G_1 \leq p^\alpha G$, $|G_2| \leq |G/p^\alpha G|$. Suppose $|G/p^\alpha G| < |G|$. Since G_1 is a pure subgroup of G and $G_1 \subset p^\alpha G \subset pG$, G_1 is p-divisible, and $|G_1| = |G|$ since $|G_2| < |G|$. □

2.7. PROPOSITION. *If G is any infinite group and $|nG| < |G|$ for some natural number n, then G has an n-bounded direct summand of cardinality $|G|$.*

PROOF. $|G/G[n]| = |nG| < |G|$, hence $G = G_1 \oplus G_2$, $G_1 \leq G[n]$ and $|G_2| \leq |nG| \cdot \aleph_0$. If nG is finite, then G is bounded and the claim follows easily. If nG is infinite, then $|G_2| < |G|$ hence $|G_1| = |G|$. □

We conclude with the proof of the proposition of the Introduction. Let L be a subgroup of H. Put $G = H$ and $K = L$. By hypothesis, $H = G = H_1 \oplus G_2$ with $L = H \cap K \subset G_2$ and $|G_2| \leq |K| \cdot \aleph_0 = |L| \cdot \aleph_0$. So H is Fuchs 5. □

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