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MODULES WITH ARTINIAN PRIME FACTORS

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ABSTRACT. An R-module M has Artinian prime factors if M/PM is an Artinian module for each prime ideal P of R. For commutative rings R it is shown that Noetherian modules with Artinian prime factors are Artinian. If R is either commutative or a von Neumann regular V-ring then the endomorphism ring of a module with Artinian prime factors is a strongly π-regular ring.

A ring R with 1 is *left π-regular* if for each $a \in R$ there is an integer $n \geqslant 1$ and $b \in R$ such that $a^n = a^{n+1}b$. Right π-regular is defined in the obvious way, however a recent result of F. Dischinger [5] asserts the equivalence of the two concepts. A ring R is *π-regular* if for any $a \in R$ there is an integer $n \geqslant 1$ and $b \in R$ such that $a^n = a^n b a^n$. Any left π-regular ring is π-regular but not conversely. Because of this, we say that R is *strongly π-regular* if it is left (or right) π-regular.

In [2, Theorem 2.5] it was established that if R is a (von Neumann) regular ring whose primitive factor rings are Artinian and if M is a finitely generated R-module then the endomorphism ring $\mathrm{End}_R(M)$ of M is a strongly π-regular ring. Curiously enough, the same is not true for finitely generated modules over strongly π-regular rings, as Example 3.1 of [2] shows. Obviously, it also fails for arbitrary regular rings. These observations lead one to consider conditions on finitely generated modules which ensure that the endomorphism ring is strongly π-regular. A natural one seems to be that of having Artinian prime factors. In fact, we establish that such modules have strongly π-regular endomorphism ring whenever the base ring is either commutative or a regular V-ring.

Consider a finitely generated module M over a ring R. If R is commutative then R is π-regular if and only if its prime ideals are maximal [11]. Accordingly, when R is commutative and π-regular, M/PM is an Artinian module for all primes P. This observation serves as a starting point.

THEOREM 1. *Suppose R is a commutative ring. For a finitely generated R-module M the following conditions are equivalent.*

(a) *M has Artinian prime factors.*

(b) $S = \mathrm{End}_R(M)$ *is a strongly π-regular ring.*

(c) $R/\mathrm{Ann}_R(M)$ *is a π-regular ring.*

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PROOF. (b) $\Rightarrow$ (c): Since M is finitely generated, S satisfies a polynomial identity. Hence each prime ideal of S is a maximal ideal (see e.g. [**1**]). Now $R' = R/\mathrm{Ann}_R(M)$ embeds in the center of S and S is an integral extension of R' [**10**]. It follows that prime ideals of R' are maximal ideals [**3**] and hence R' is π-regular.

(c) $\Rightarrow$ (a): This is clear.

(a) $\Rightarrow$ (b): By [**2**, Proposition 2.3], S is strongly π-regular if and only if for each $\alpha \in S$ there is an integer $t \geqslant 1$ such that $M = \mathrm{Ker}\,\alpha^t \oplus M\alpha^t$. Thus let $\alpha \in S$. Our first step is to show that there is an integer $t \geqslant 1$ such that $M\alpha^t = M\alpha^{t+1}$. Assume that no such integer exists. Because M is finitely generated, there is an ideal P of R which is maximal among those ideals I of R having the property that $M\alpha^k \not\subseteq M\alpha^{k+1} + IM$ for all integers $k \geqslant 1$. We claim that P is a prime ideal of R. Thus suppose A and B are ideals of R properly containing P and such that $AB \subseteq P$. We then have integers m, n such that $M\alpha^m \subseteq M\alpha^{m+1} + AM$ and $M\alpha^n \subseteq M\alpha^{n+1} + BM$. The second of these inclusions gives us $AM\alpha^n \subseteq AM\alpha^{n+1} + PM$, since $AB \subseteq P$. Hence $AM\alpha^{n+1} \subseteq AM\alpha^{n+2} + PM$, giving $AM\alpha^n \subseteq AM\alpha^{n+2} + PM$. Continuing we arrive at $AM\alpha^n \subseteq AM\alpha^{n+m+1} + PM$ and therefore $AM\alpha^n \subseteq M\alpha^{n+m+1} + PM$. Using this we then get $M\alpha^{n+m} = (M\alpha^m)\alpha^n \subseteq (M\alpha^{m+1} + AM)\alpha^n \subseteq M\alpha^{n+m+1} + PM$, which contradicts the choice of P. Thus P is a prime ideal as claimed. By assumption, M/PM is an Artinian module. But then the sequence of submodules $M\alpha \supseteq M\alpha^2 \supseteq \cdots$ must terminate modulo PM, providing the desired contradiction. This shows then that $M\alpha^t = M\alpha^{t+1}$ for some integer $t \geqslant 1$. Now α is an onto endomorphism of the finitely generated R-module $M\alpha^t$, and so α is 1-1 on $M\alpha^t$ since R is commutative [**12**]. Then $\mathrm{Ker}\,\alpha \cap M\alpha^t = 0$ implies that $\mathrm{Ker}\,\alpha^t = \mathrm{Ker}\,\alpha^{t+1}$. It now follows easily that $M = \mathrm{Ker}\,\alpha^t \oplus M\alpha^t$, completing the proof.

Examination of the proof of the implication (a) $\Rightarrow$ (b) shows that commutativity was used only to ensure that onto endomorphisms are 1-1. It has been shown in [**2**, Theorem 2.2] that rings integral over their center and satisfying a polynomial identity have the property that onto endomorphisms of finitely generated modules are 1-1, a property which left Noetherian rings also have. Thus we are able to state the following.

THEOREM 2. *Assume R is either a PI-ring integral over its center or a left Noetherian ring. If M is a finitely generated left R-module having Artinian prime factor then* $\mathrm{End}_R(M)$ *is a strongly π-regular ring.*

In view of this theorem one might ask if any Noetherian R-module with Artinian prime factors is Artinian. The answer is no, in general. An example in [**9**, p. 66] provides us with a perfect ring D having a Noetherian non-Artinian module. Since D/P is simple Artinian for any prime ideal P, such a module must have Artinian prime factors. Before showing that the answer is affirmative when R is commutative, we note that over a semiprimary ring, all Noetherian modules are Artinian, so the answer is (trivially) yes in this case.

THEOREM 3. *If R is a commutative ring and M is a Noetherian R-module with Artinian prime factors then M is Artinian.*

PROOF. By Theorem 1, $R' = R/\mathrm{Ann}_R(M)$ is a π-regular ring. Since M is a faithful finitely generated R'-module, R' is isomorphic to a submodule of a finite direct sum of copies of M. Hence R' is a Noetherian module. Any Noetherian π-regular ring is Artinian so R' is Artinian. But then M, being a finitely generated R'-module, must be Artinian.

While it is false in general that Noetherian modules with Artinian prime factors are Artinian, the following is true.

THEOREM 4. *If M is a Noetherian R-module with Artinian prime factors then $S = \mathrm{End}_R(M)$ is semiprimary.*

PROOF. The proof of (a) $\Rightarrow$ (b) shows that S is a strongly π-regular ring. Thus each nonnil one sided ideal contains a nonzero idempotent. It follows that $J(S)$, the Jacobson radical of S, is a nil ideal. By a theorem of L. Small (see [**6**, Theorem 2.1]), nil subrings of S are nilpotent, so that $J(S)$ is a nilpotent ideal of S. Now orthogonal idempotents of $S/J(S)$ can be lifted to S. However M is Noetherian so S can have no infinite set of orthogonal idempotents. It follows then that $S/J(S)$ is a semisimple Artinian ring.

This theorem generalizes the well-known fact that the . ndomorphism ring of a Noetherian Artinian module is semiprimary.

We now turn to a result which covers [**2**, Theorem 2.3]. Recall that a (left) *V-ring* is a ring all of whose simple left modules are injective. For the salient features of V-rings we refer the reader to [**4**, Chapter 5].

THEOREM 5. *Assume R is a regular V-ring. If M is a finitely generated R-module with Artinian prime factors then $S = \mathrm{End}_R(M)$ is a strongly π-regular ring.*

PROOF. Let $\alpha \in S$. As in the proof of Theorem 1, there is an integer $t \geqslant 1$ such that $M\alpha^t = M\alpha^{t+1}$. Suppose $x \in \mathrm{Ker}\,\alpha^{t+1}$; if $u = x\alpha^t \neq 0$, then there is an ideal P of R maximal among those ideals I of R for which $u \notin IM$. Hence $u \in AM$ for all ideals A of R properly containing P. If P is not a prime ideal then there are ideals A and B of R properly containing P for which $AB \subseteq P$. Then $u \in BM$ so that $Au \subseteq ABM \subseteq PM$. Since we also have $u \in AM$ we can write $u = \Sigma a_i m_i$ where $a_i \in A$, $m_i \in M$. Because R is a regular ring there is an idempotent $e \in A$ such that $ea_i = a_i$ for each i. But then $u = eu \in Au \subseteq PM$, a contradiction. Thus P must be a prime ideal and the module M/PM is Artinian. Because R is a V-ring and M/PM has finitely generated essential socle, we infer that M/PM is completely reducible and hence Noetherian. Then α induces $\beta \in \mathrm{End}_R(M/PM)$ and $(M/PM)\beta^t = (M/PM)\beta^{t+1}$ and this yields $\mathrm{Ker}\,\beta^t = \mathrm{Ker}\,\beta^{t+1}$. But then $x\alpha^t \in PM$, which is the desired contradiction. It now follows that $\mathrm{Ker}\,\alpha^t = \mathrm{Ker}\,\alpha^{t+1}$, $M = M\alpha^t \oplus \mathrm{Ker}\,\alpha^t$, and so S is strongly π-regular.

COROLLARY 5 [**2**, THEOREM 2.3]. *If R is a regular ring whose primitive factor rings are Artinian then $\mathrm{End}_R(M)$ is strongly π-regular for any finitely generated R-module M.*

PROOF. It is enough to note that (i) R is a V-ring, and (ii) prime factor rings of R are Artinian. That (i) holds follows from [**4**, Corollary 5.13] while [**8**, Theorem 3, p. 239] guarantees (ii).

It is straightforward to see that a finitely generated projective Artinian module over a semiprime ring is completely reducible. Thus the proof of Theorem 4 can be used to prove the next result.

THEOREM 6. *Let R be a regular ring and M a finitely generated R-module. If M/PM is a projective Artinian R/P-module for each prime ideal P of R then* $\mathrm{End}_R(M)$ *is strongly π-regular.*

In the first version of this article we asked whether or not a finitely generated Artinian module over a regular ring is Noetherian. An affirmative answer would then imply the statement,

over any regular ring, finitely generated modules with Artinian prime factors have a strongly π-regular endomorphism ring. (*)

Recently, K. Goodearl has constructed examples of cyclic Artinian non-Noetherian modules as well as Noetherian non-Artinian modules over regular rings [7]. Thus our original question has a negative answer. However the validity of (*) still remains open, and would be true should the following question have a positive response. If M is a finitely generated Artinian module over a regular ring, is every onto endomorphism of M also 1-1?

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