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FRATTINI EMBEDDINGS OF NORMAL SUBGROUPS

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ABSTRACT. If N is a normal subgroup of the finite group G, there is a split extension X of an elementary abelian group by G such that $N \trianglelefteq X$ and $N \cap \Phi(G) = \Phi(X)$.

R. B. J. T. Allenby proved in [1] that if G is a finite group and if N is a normal subgroup contained in the Frattini group $\Phi(G)$, then there is a finite group X with $N = \Phi(X)$. His ingenious argument depends on properties of subgroups of free products, and uses infinite groups in an essential way. In this note we show how such an X can be built up from the inside, using an elementary construction which makes calculations in X easy and which retains in X some of the qualities of G. The following theorem lists the main properties of X and gives Allenby's conclusion in case $N \leqslant \Phi(G)$.

THEOREM. *Suppose that N is a normal subgroup of the finite group G. Then there is a finite group X such that*

(a) *X is a split extension of an elementary abelian group by G,*

(b) *$N \trianglelefteq X$,*

(c) *$N \cap \Phi(G) = \Phi(X)$, and*

(d) *X acts on N by conjugation as G acts on N.*

To build X, let p be a prime not dividing $|G|$, and let B be a $\mathbf{Z}_p[G/N]$-module faithful for G/N. Let H be the semidirect product $B](G/N)$, and let $X = B]G$ with $N = C_G(B)$ and the given action of G/N on B. Then $N \trianglelefteq X$ and $X/N \simeq H$.

By Maschke's Theorem, $B \leqslant \mathrm{Soc}(H)$. Hence, $\Phi(H) \leqslant \mathrm{Fit}(H) \leqslant C_H(B) = B$. If A is a minimal normal subgroup of H contained in B, then $B = A \times D$ for some D normal in H, A is complemented in H by the maximal subgroup $(G/N)D$, and so $A \nleqslant \Phi(H)$. Thus $\Phi(H) = 1$. Since $\Phi(X)N/N \leqslant \Phi(X/N) \simeq \Phi(H)$, $\Phi(X) \leqslant N$. Since $\Phi(X)B/B \leqslant \Phi(X/B) = \Phi(G)B/B$, $\Phi(X) \leqslant N \cap (\Phi(G)B) = N \cap \Phi(G)$. On the other hand, $X = BG = N_G(N \cap \Phi(G))$, so, by Theorem 7.3.16 of [2], $N \cap \Phi(G) \leqslant \Phi(X)$, and the proof is complete.

If G is solvable, then so is the group X just constructed. At the expense of solvability, the argument just given can be modified to produce a group X with $G \leqslant X$, $N \trianglelefteq X$, $N \cap \Phi(G) = \Phi(X)$ and $N \cap \mathrm{Fit}(G) = \mathrm{Fit}(X)$. Instead of the choice used above, let S be a simple group, and let H be the wreath product $S \wr (G/N)$. Let B be the base group of H, and let $X = B]G$ with $N = C_X(B)$, and

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with the action of G/N on B in X the same as in H. Then $\Phi(H) \leq \text{Fit}(H) \leq C_H(B) = Z(B) = 1$. The rest of the argument that $N \cap \Phi(G) = \Phi(X)$ goes as before. Since $N \cap \text{Fit}(G) \trianglelefteq BG = X$, $N \cap \text{Fit}(G) \leq \text{Fit}(X)$. Finally, since $\text{Fit}(X)N/N \leq \text{Fit}(H) = 1$, $\text{Fit}(X) = N \cap \text{Fit}(X) \leq \text{Fit}(G)$.

Let $\text{Fit}_1(G) = \text{Fit}(G)$ and $\Phi_1(G) = \Phi(G)$, and for $k \geq 1$ define $\text{Fit}_{k+1}(G)/\text{Fit}_k(G) = \text{Fit}(G/\text{Fit}(G))$ and $\Phi_{k+1}(G)/\text{Fit}_k(G) = \Phi(G/\text{Fit}_k(G))$. One can show that the second construction of X gives $N \cap \text{Fit}_k(G) = \text{Fit}_k(X)$ for $k \geq 1$, but it does not ensure that $N \cap \Phi_k(G) = \Phi_k(X)$ for $k > 1$. For example, let $G = (\langle a \rangle]\langle b \rangle) \times \langle c \rangle$, with $|a| = 5$, $|b| = 4$, $|c| = 2$, $a^b = a^2$, and let $N = \langle a, b^2c \rangle$. Then $\Phi_2(X) < \Phi_2(G) \cap N$.

References

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