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GENERALIZATIONS OF A THEOREM OF MUTYLIN

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ABSTRACT. We generalize Mutylin's theorem that the only complete, locally bounded, additively generated topological fields are **R** and **C** by showing: (1) the only complete, locally bounded, additively generated topological division rings with left bounded commutator subgroup are **R**, **C**, and **H**; (2) a commutative, Hausdorff topological ring A with identity is a Banach algebra over **R**, equipped with the absolute value $|..|^p$ for some $p \in (0, 1]$, if (and only if) A is complete, locally bounded, additively generated, and possesses an invertible topological nilpotent.

In 1968, Mutylin [**8**, Theorem 1] proved that the only complete, additively generated, locally bounded fields were **R** and **C** (a topological ring is *additively generated* if it contains no proper open additive subgroups; any connected ring is additively generated). This theorem implies, of course, the commutative part of Pontrjagin's classification of locally compact, connected division rings. Mutylin's proof depended essentially on the Jordan Curve Theorem for the plane.

Here we shall generalize Mutylin's theorem in two ways, and our proofs will not depend on any deep properties of **C** other than the fact that **C** is locally compact and has roots of unity of all orders, a fact needed in elementary proofs of the Gelfand-Mazur theorem and Ostrowski's theorem.

Convexity is unneeded in proofs of basic theorems concerning general Banach algebras. More precisely, basic theorems, such as the Gelfand-Mazur theorem, hold for Banach algebras over **R** or **C** equipped with the absolute value $|..|^p$ for some $p \in (0, 1]$, as Żelazko has shown [**12**].

A *seminorm* on a ring K is a function N from K to **R** such that for all $x, y \in K$, $N(x) \geqslant 0$, $N(-x) = N(x)$, $N(xy) \leqslant N(x)N(y)$, and $N(x + y) \leqslant N(x) + N(y)$. The null space $N^{-1}(0)$ of a seminorm N is an ideal, and N is a *norm* if its null space is (0). An *absolute semivalue* on a ring K with identity is a seminorm A such that $A(1) = 1$ and $A(xy) = A(x)A(y)$ for all $x, y \in K$. The null space of an absolute semivalue A is a prime ideal, and A is an *absolute value* if its null space is (0).

If N is a seminorm on K, we define N_s by $N_s(x) = \lim_{n\to\infty} N(x^n)^{1/n}$. Then x is a topological nilpotent (that is, $\lim_{n\to\infty} x^n = 0$) if and only if $N_s(x) < 1$. If K is commutative, or if K is a division ring whose commutator subgroup Γ is norm-bounded, then N_s is a seminorm. Indeed, in the latter case, to show that $N_s(x + y) \leqslant N_s(x) + N_s(y)$, one needs to modify the usual proof in the commutative case by

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using the variant of the Binomial Theorem that states that $(x + y)^n = \sum_{k=0}^{n} s_{n,k} x^{n-k} y^k$ where each $s_{n,k}$ is the sum of $\binom{n}{k}$ members of Γ and $s_{n,0} = s_{n,n} = 1$; and to prove that $N_s(xy) \leqslant N_s(x)N_s(y)$, one uses the fact that $(xy)^n = g_n x^n y^n$ where $g_n \in \Gamma$. A seminorm N is *spectral* if $N = N_s$, or equivalently, if $N(x^n) = N(x)^n$ for all $x \in K$ and all $n \geqslant 1$. If N_s is a seminorm, it is a spectral seminorm.

THEOREM 1. *Let K be a division ring equipped with a Hausdorff ring topology. The following conditions are sufficient (and necessary) for K to be topologically isomorphic to* $\mathbf{R}$, $\mathbf{C}$, *or the topological division ring* $\mathbf{H}$ *of quaternions*:

1. *K is complete.*
2. *K is locally bounded.*
3. *K is additively generated.*
4. *The commutator subgroup Γ of K is left bounded.*

PROOF. K contains a nonzero topological nilpotent [4, Exercise 21, p. 121], and there is a bounded neighborhood U of zero such that $UU \subseteq U$ [4, Exercise 20d), p. 121]. Let $V = \Gamma U$. Then V is a left bounded neighborhood of zero such that $aV = Va$ for all $a \in K$. By a generalization of a theorem of Cohn [6, Theorem 6.1] due to Lipkina [7], the topology of K is given by a norm N, and as the topology of K is not discrete, Γ is norm bounded. Therefore N_s is a spectral norm (as K is a division ring). By a theorem of Aurora [3, Theorem 1], $N_s = \sup_{c \in K^*} A_c$, where for each $c \in K^*$, A_c is an absolute value (as K is a division ring) satisfying $A_c \leqslant N_s$ and $A_c(c) = N_s(c)$. For each $c \in K^*$, $\{x \in K\colon A_c(x) \leqslant 1\}$ is a neighborhood of zero and therefore is not an additive subgroup, so A_c is archimedean. Consequently, by Ostrowski's theorem [5, Theorem 2, p. 131], there exist an isomorphism σ_c from K onto a division subring of $\mathbf{H}$ and a number $p_c \in (0, 1]$ such that $A_c(x) = |\sigma_c(x)|^{p_c}$ for all $x \in K$. Therefore K has characteristic zero and thus contains the rational field $\mathbf{Q}$. Let $r \in \mathbf{Q}^*$. Since $|r|^{p_r} = A_r(r) = N_s(r)$, $|r| < 1$ if and only if $N_s(r) < 1$, that is, if and only if r is a topological nilpotent. Consequently, the topology induced on $\mathbf{Q}$ is not discrete and also is not the p-adic topology for any prime p (since the topological nilpotents for the p-adic topology form a nonzero additive subgroup of $\mathbf{Q}$). Therefore by 1 and [10, Corollary 2], the closure of $\mathbf{Q}$ in K is the real field $\mathbf{R}$.

Since K is locally bounded as a ring, it is *a fortiori* locally bounded as a vector space over its subfield $\mathbf{R}$. Consequently, by a theorem of Aoki [1], rediscovered by Rolewicz [9], there is a vector space norm M on the $\mathbf{R}$-vector space K relative to the absolute value $|..|^p$ on $\mathbf{R}$ for some $p \in (0, 1]$ that defines the topology of K. As multiplication is jointly continuous in both variables, there is an equivalent Banach algebra norm $\|..\|$ (defined by $\|x\| = t^{-2}M(x)$, where $t > 0$ is such that $M(x) \leqslant t$ and $M(y) \leqslant t$ imply $M(xy) \leqslant 1$). Thus K is a normed division algebra over $\mathbf{R}$, equipped with $|..|^p$. By the Gelfand-Mazur theorem, K is topologically isomorphic to $\mathbf{R}$, $\mathbf{C}$, or $\mathbf{H}$.

A similar proof characterizes those commutative topological rings with identity that are real Banach algebras:

THEOREM 2. *Let A be a Hausdorff commutative topological ring with identity e. The following conditions are sufficient (and necessary) for A to be a Banach algebra over* $\mathbf{R}$, *equipped with the absolute value $|..|^p$ for some $p \in (0, 1]$:*

1. *A is complete.*
2. *A is locally bounded.*
3. *A is additively generated.*
4. *A contains an invertible topological nilpotent.*

PROOF. The conditions are necessary, for $r \cdot e$ is an invertible topological nilpotent for any nonzero $r \in \mathbf{R}$ such that $|r| < 1$, and A is connected and hence additively generated.

Sufficiency. By 2, 4, and a generalization of Cohn's theorem [**2**, Corollary of Theorem 3], [**11**, Theorem 4], the topology of A is given by a norm N. Let J be the null space of the associated spectral seminorm N_s. By Aurora's theorem [**3**, Theorem 1], $N_s = \sup_{c \in A \setminus J} A_c$, where for each $c \in A \setminus J$, A_c is an absolute semivalue satisfying $A_c \leqslant N_s$ and $A_c(c) = N_s(c)$. Let J_c be the null space of A_c, a prime ideal of A, and let K_c be the quotient field of A/J_c. In a natural way A_c induces an absolute value on A/J_c, which has a unique extension to an absolute value A_c' on K_c. If the closed unit ball B_c of A_c' were an additive subgroup, then $B_c \cap (A/J_c)$ would also be an additive subgroup; its inverse image under the canonical epimorphism φ_c from A to A/J_c, namely, $\{x \in A : A_c(x) \leqslant 1\}$, would then be an additive subgroup, which is impossible by 3, for as $A_c \leqslant N_s \leqslant N$, that set is a neighborhood of zero. Therefore A_c' is an archimedean absolute value, so A/J_c has characteristic zero and hence A does also.

Next, we shall show that for each integer $m > 1$, $m \cdot e$ is invertible. In the contrary case, there would be a proper ideal and hence a maximal ideal M containing $m \cdot e$. As A is a complete normed ring, the set of its invertible elements is open, so M would be a closed ideal. Then A/M would clearly satisfy conditions 1–4 and hence, by what we have just proved, would have characteristic zero, a contradiction, since $m \cdot e \in M$.

Therefore A contains the rational field $\mathbf{Q}$. By Ostrowski's theorem, there exist an isomorphism σ_c from K_c onto a subfield of $\mathbf{C}$ and a number $p_c \in (0, 1]$ such that $A_c'(z) = |\sigma_c(z)|^{p_c}$ for all $z \in K_c$. Let $u_c = \sigma_c \circ \varphi_c$, a nonzero homomorphism from A to $\mathbf{C}$; then $A_c(x) = |u_c(x)|^{p_c}$ for all $x \in A$, and in particular, $A_c(r) = |r|^{p_c}$ for all $r \in \mathbf{Q}$. Proceeding as in the proof of Theorem 1, we conclude that A contains $\mathbf{R}$ and that A is a Banach algebra over $\mathbf{R}$ relative to the absolute value $|..|^p$ for some $p \in (0, 1]$.

COROLLARY. *A commutative Hausdorff topological ring A with identity e is a Banach algebra over* $\mathbf{C}$, *equipped with the absolute value $|..|^p$ for some $p \in (0, 1]$, if and only if A satisfies* 1–4 *of Theorem* 2 *and, in addition, possesses an element i such that $i^2 = -e$.*

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