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COMPLETE INTERSECTIONS IN C^n AND R^{2n}

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ABSTRACT. When $\mathbf{C}^n$ is identified with $\mathbf{R}^{2n}$ in the usual way, algebraic varieties over the complex numbers give rise to varieties over the reals. We ask when a (strict) complete intersection in $\mathbf{C}^n$ yields a (strict) complete intersection in $\mathbf{R}^{2n}$. If the original variety V is connected, a necessary and sufficient condition that its image be a complete intersection is that V be irreducible. We give examples that show that without the connectedness assumption the conclusion is false.

In the course of proving this result we give an algebraic analogue of a result by Ephraim on germs of complex and the corresponding real analytic varieties. As our methods apply to varieties over the algebraic closure of an arbitrary real closed field the paper is written in this more general setting.

1. Preliminaries. Throughout this paper R will denote a real closed field and C its algebraic closure. Thus $C = R[i]$ where $i^2 = -1$.

By an algebraic variety in R^m (resp. in C^m) we mean the zero set of a collection of polynomial equations. We recall some definitions and results from the theory of real varieties. One can consult [1] or [3] for more details and proofs of these assertions.

Let $P = R[x_1, \ldots, x_m]$ be a polynomial algebra over R. Let $\sigma(P) = \{f \in P | f = g_1^2 + \cdots + g_r^2$ for some $g_1, \ldots, g_r \in P\}$. Given an ideal $\mathfrak{a} \subset P$ define a set $S_\sigma(\mathfrak{a})$ by

$$S_\sigma(\mathfrak{a}) = \{f \in P | [f^2 + \sigma(P)] \cap \mathfrak{a} \neq \varnothing\}.$$

DEFINITIONS 1.1. (i) The ideal $\mathfrak{a}$ is a real ideal if $S_\sigma(\mathfrak{a}) = \mathfrak{a}$.

(ii) A real prime of P is a prime ideal $\mathfrak{p}$ of P such that $\mathfrak{p}$ is a real ideal.

(iii) The real radical of $\mathfrak{a}$, denoted $\operatorname{rlrad}(\mathfrak{a})$, is the intersection of all real primes of P which contain $\mathfrak{a}$.

PROPOSITION 1.2. *Let P, $\mathfrak{a}$ be as above. Then*

(i) $S_\sigma(\mathfrak{a})$ *is an ideal containing* $\mathfrak{a}$.

(ii) $\operatorname{rlrad}(\mathfrak{a}) = \operatorname{rad}(S_\sigma(\mathfrak{a}))$ *is a real ideal.*

(iii) *The minimal real primes of* $\mathfrak{a}$ *are the minimal primes of* $\operatorname{rlrad}(\mathfrak{a})$.

(iv) $\operatorname{rlrad}(\mathfrak{a}) = \operatorname{rad}(\mathfrak{a})$ *if and only if every minimal prime of* $\mathfrak{a}$ *is real.*

(v) *If $\mathfrak{p}$ is a real prime of P and $B = P \otimes_R C$ then $\mathfrak{p}B$ is a prime ideal of B.*

If $E \subset P$ is a subset let $V(E) = \{x \in R^m | f(x) = 0$ for all $f \in E\}$. If $X \subset R^m$ is a subset let $I(X) = \{f \in P | f|_X \equiv 0\}$. We now state the real nullstellensatz.

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THEOREM 1.3 (DUBOIS [2], RISLER [7]). *If* $\mathfrak{a} \subset P$ *is an ideal, then* $I(V(\mathfrak{a})) = \text{rlrad}(\mathfrak{a})$.

We now wish to describe the complexification of a real variety. The situation is entirely analogous to what happens over the real and complex numbers. In this setting the complexification was described by Whitney in [8].

Let $V \subset R^m$ be a variety. Regarding R^m as a subset of C^m we wish to describe the complexification of V, denoted V^*. V^* is the (unique) smallest complex variety in C^m containing V as its real points. As above, let $I = I(V) = \{f \in P \mid f|_V \equiv 0\}$. Let $I^* = I(V^*) = \{f \in B \mid f|_{V^*} \equiv 0\}$. Then $I^* = I \otimes_R C$. If $\mathfrak{p}_1, \ldots, \mathfrak{p}_l$ are the minimal (real) primes of I, then $\mathfrak{p}_1^* = \mathfrak{p}_1 \otimes C, \ldots, \mathfrak{p}_l^* = \mathfrak{p}_l \otimes C$ are the minimal primes of I^*. We note that V and V^* have the same number of irreducible components.

For an irreducible variety V in R^m we define the dimension of V over R, denoted $\dim_R V$, by $\dim_R V = \text{trdeg}_R P/I(V)$. For an arbitrary variety in R^m the dimension is the maximal dimension of an irreducible component. Hence $\dim_R V = \dim_C V^*$.

Viewing V as a topological space, we have dimension and codimension defined in terms of chains of irreducible closed subsets.

In lieu of the following lemma we have

$$\dim_R V = \dim V, \qquad \text{codim}(V, R^m) = \text{ht}\, I(V).$$

LEMMA 1.1. *Let* $\mathfrak{B} \subset R[x_1, \ldots, x_m]$ *be a real prime of dimension* d. *Then there exists a chain of distinct real primes*

$$(0) = \mathfrak{B}_0 < \cdots < \mathfrak{B}_{m-d} = \mathfrak{B} < \mathfrak{B}_{m-d+1} < \cdots < \mathfrak{B}_m.$$

PROOF. Since $\mathfrak{B}$ is a real prime we know there exists a simple point $p \in R^m$ for $\mathfrak{B}$ (see [1, p. 49, Theorem 4.7]). Hence there exist $f_1, \ldots, f_{m-d}$ in $\mathfrak{B}$ such that the Jacobian matrix of $(f_1, \ldots, f_{m-d})$ at p has rank $m - d$.

Let $\mathfrak{m} \subset P$ denote the real maximal ideal corresponding to p. Thus $(f_1, \ldots, f_{m-d})P_\mathfrak{m} = \mathfrak{B}_\mathfrak{m}$ and there exist $f_{m-d+1}, \ldots, f_m$ in $\mathfrak{m}P_\mathfrak{m}$ such that $f_1, \ldots, f_m$ is a regular system of parameters. Let $\mathfrak{B}_i$ be the prime of P such that $(\mathfrak{B}_i)_\mathfrak{m} = (f_1, \ldots, f_i)P_\mathfrak{m}$ for $i = 1, \ldots, m$. Then $\mathfrak{B}_i$ is a prime of height i (see [6, p. 121, Theorem 3.6]). Since $p \in R^m$ is a simple point for each $\mathfrak{B}_i$ this yields the required chain of real primes.

2. Main results. Regard C^n as R^{2n} having coordinates $x_1, y_1, \ldots, x_n, y_n$. If $X \subset C^n$ let X' denote the corresponding subset of R^{2n} and similarly if $p \in C^n$ let p' denote the corresponding point in R^{2n}.

Let $C[z_1, \ldots, z_n] = C[z]$ and $C[x_1, y_1, \ldots, x_n, y_n] = C[x, y]$ be polynomial algebras over C. Identifying $C[x, y]$ and $R[x, y] \otimes_R C$ every polynomial $f \in C[x, y]$ can be written uniquely in the form $f = g + ih$, where $g, h \in R[x, y]$. In this representation g (resp. h) is called the real (resp. imaginary) part of f.

Define a C-algebra map $\varphi\colon C[z] \to C[x, y]$ by

$$\varphi(z_\lambda) = x_\lambda + iy_\lambda, \qquad \lambda = 1, \ldots, n.$$

If $f \in C[z]$ let $g, h \in R[x, y]$ be the real and imaginary parts respectively of $\varphi(f)$. Thus for a point $p \in C^n$ we have $f(p) = 0$ iff $g(p') = 0 = h(p')$ where $p' \in R^{2n}$.

(2.0) Let $V \subset C^n$ be the zero set of polynomials $f_1, \ldots, f_s$ and let

$$I_C(V) = \{f \in C[z] \mid f|_V \equiv 0\},$$
$$I_R(V') = \{f \in R[x, y] \mid f|_{V'} \equiv 0\}.$$

Let g_λ and h_λ denote the real and imaginary parts of $\varphi(f_\lambda)$ respectively, for $\lambda = 1, \ldots, s$. Then by the Hilbert and real nullenstellensatzen we have

$$I_C(V) = \mathrm{rad}(f_1, \ldots, f_s),$$
$$I_R(V') = \mathrm{rlrad}(g_1, \ldots, g_s, h_1, \ldots, h_s).$$

We now give an analogue of a theorem by Ephraim [**4**, Theorem 2.1] on germs of complex and their associated real analytic varieties. Our method of proof in one direction follows Ephraim's argument. Before we state this result we establish some helpful notation.

For a polynomial $f \in C[w_1, \ldots, w_m] = C[w]$ let $\tilde{f} = \sum \bar{a}_\alpha w^\alpha$ where $f = \sum a_\alpha w^\alpha$, the index α runs through elements of $\mathbf{N}^m$ and the bar denotes "complex" conjugation. For a subset $X \subset C^m$ let $\overline{X} = \{\bar{p} \mid p \in X\}$ denote the conjugate subset.

PROPOSITION 2.1. *With notation as in* (2.0), *let $V \subset C^n$ be a variety with defining ideal $I = I_C(V) = (f_1, \ldots, f_s)$. Then $I' = I_R(V') = (g_1, \ldots, g_s, h_1, \ldots, h_s)$ if and only if V is irreducible.*

PROOF. Let $\mathfrak{a} = (g_1, \ldots, g_s, h_1, \ldots, h_s)$ so that $I' = \mathrm{rlrad}(\mathfrak{a})$.

Suppose that V is irreducible so that I is prime. We claim that $J = \mathfrak{a} \otimes_R C$ is prime (and hence $\mathfrak{a}$ is prime).

We first note that $J = (g_1 + ih_1, \ldots, g_s + ih_s, g_1 - ih_1, \ldots, g_s - ih_s)$.

Define a C-algebra isomorphism ψ: $C[u_1, \ldots, u_n] \otimes_C C[v_1, \ldots, v_n] \to C[x_1, y_1, \ldots, x_n, y_n]$ by

$$\psi(u_\alpha) = x_\alpha + iy_\alpha, \qquad \alpha = 1, \ldots, n,$$
$$\psi(v_\beta) = x_\beta - iy_\beta, \qquad \beta = 1, \ldots, n.$$

Then

$$\psi(f_\lambda(u_1, \ldots, u_n)) = g_\lambda + ih_\lambda, \qquad \lambda = 1, \ldots, s$$

and

$$\psi\big(\tilde{f}_\rho(v_1, \ldots, v_n)\big) = g_\rho - ih_\rho, \qquad \rho = 1, \ldots, s.$$

Thus $\psi^{-1}(J) = (f_1(u), \ldots, f_s(u), \tilde{f}_1(v), \ldots, \tilde{f}_s(v))$ is prime as the latter is the ideal of the irreducible variety $V \times \overline{V}$ in C^{2n}.

Hence $\mathfrak{a}$ is prime. Since $I' = \mathrm{rlrad}(\mathfrak{a})$ it suffices to see that $\mathfrak{a}$ is a real prime. It thus suffices to exhibit a real simple point for $\mathfrak{a}$ (see [**1**, p. 49, Theorem 4.7]).

Let $d = \dim V = \mathrm{trdeg}_C(C[z]/I)$. Then

$$\mathrm{trdeg}_C\big(C[x, y]/\mathfrak{a} \otimes C\big) = 2d \text{ so that } \mathrm{trdeg}_R\big(R[x, y]/\mathfrak{a}\big) = 2d.$$

Let $p \in V$ be a simple point so that the Jacobian matrix of $(f_1, \ldots, f_s)$ at p has rank $n - d$. As we have

$$\frac{\partial \tilde{f}}{\partial z_k}(\bar{p}) = \overline{\frac{\partial f}{\partial z_k}(p)}, \qquad k = 1, \ldots, n,$$

we see that $\bar{p}$ is a simple point for $\bar{V}$ and hence $(p, \bar{p})$ is a simple point for $V \times \bar{V}$. In lieu of the isomorphism ψ above we conclude that the Jacobian matrix of $(g_1 + ih_1, \ldots, g_s + ih_s, g_1 - ih_1, \ldots, g_s - ih_s)$ at p' has rank $2n - 2d$. As this rank is independent of the choice of ideal generators we have the Jacobian matrix of $(g_1, \ldots, g_s, h_1, \ldots, h_s)$ at p' has rank $2n - 2d$. Hence p' is a real simple point for $\mathfrak{a}$ and $\mathfrak{a}$ is a real prime.

Conversely, assume $I' = (g_1, \ldots, g_s, h_1, \ldots, h_s)$. Then $I' \otimes_R C = (g_1 + ih_1, \ldots, g_s + ih_s, g_1 - ih_1, \ldots, g_s - ih_s)$ is the ideal defining the complexification $(V')^*$ of V'. Hence $(V')^*$ is isomorphic to $V \times \bar{V}$ as above.

Let l denote the number of irreducible components of V. By the first half of the proof, V' has l components. Then $(V')^*$ has l components while $V \times \bar{V}$ has l^2 components. Thus $l = l^2$ so that $l = 1$.

REMARK 2.2. We note that as in the proof of (2.1) if $V \subset C^n$ is an irreducible variety then $V' \subset R^{2n}$ is also irreducible. More generally the irreducible components of V correspond to those of V', $\dim(V') = 2\dim(V)$ and $\operatorname{codim}(V', R^{2n}) = 2\operatorname{codim}(V, C^n)$.

Before we state our main result we recall some definitions and a result by Hartshorne on complete intersections and connectedness. The reader is referred to Hartshorne's paper [5] for a proof of Proposition 2.4.

DEFINITION 2.3. A variety V in R^m (resp. in C^m) is said to be a strict complete intersection (s.c.i. for short) if its ideal of definition is generated by s polynomials where s is the codimension of V in R^m (resp. in C^m).

We will say a variety V in R^m is a weak complete intersection (w.c.i. for short) if its ideal of definition is the nilradical of an ideal generated by s polynomials where $s = \operatorname{codim}(V, R^m)$.

PROPOSITION 2.4 (HARTSHORNE). *Let X be a connected, locally noetherian scheme and let Y be a closed subset of X such that for each $y \in Y$ the local ring $\mathcal{O}_{X,y}$ has depth at least* 2. *Then $X - Y$ is connected.*

In what follows, by component we mean irreducible component. Our main result is

THEOREM 2.5. *Suppose a connected variety $V \subset C^n$ is an s.c.i. Then $V' \subset R^{2n}$ is a w.c.i. if and only if V' is an s.c.i. if and only if V is irreducible.*

PROOF. If V is irreducible then (2.1) entails V' is an s.c.i. and hence a w.c.i.

So suppose V' is a w.c.i. Notice that V' is also connected. If not, the components of V' (and hence of V) could be divided into two disjoint classes, contradicting the connectedness of V.

Let $U \subset C^{2n}$ denote the complexification of V' so that U is also connected. We claim that U in C^{2n} is a w.c.i. For let $I = I_R(V') = \operatorname{rad}(k_1, \ldots, k_{2s})$ where $s = \operatorname{codim}(V, C^n)$. As I is a real ideal we know that every minimal prime of I is real so that if $I = \mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_l$ then $I^* = \mathfrak{p}_1^* \cap \cdots \cap \mathfrak{p}_l^*$. Now $R[x, y] \subset C[x, y]$ is an integral extension of domains and the former is integrally closed in its quotient field, so that the going-down property is in effect. Hence $\mathfrak{p}_1^*, \ldots, \mathfrak{p}_l^*$ are the minimal primes of the ideal in $C[x, y]$ generated by $k_1, \ldots, k_{2s}$ so that $I_C(U)$ is the radical of $(k_1, \ldots, k_{2s})C[x, y]$. Hence U is a w.c.i.

We will argue using the topological properties of $\operatorname{Spec}(C[x, y]/I_C(U))$ and since this scheme is homeomorphic to $\operatorname{Spec}(C[x, y]/(k_1, \ldots, k_{2s}))$ we may and shall assume that $I^* = I_C(U) = (k_1, \ldots, k_{2s})$. (Our ideals are now taken in $C[x, y]$.)

Since $\operatorname{ht} I^* = \operatorname{codim}(U, C^{2n}) = \operatorname{codim}(V', R^{2n}) = 2s$ we see that $k_1, \ldots, k_{2s}$ is a regular sequence in the Cohen-Macaulay ring $C[x, y]$. Hence $A = C[x, y]/I^*$ is again Cohen-Macaulay (see [**6**, (16.B)–(16.D)]). Thus if $Y \subset X = \operatorname{Spec}(A)$ is closed of codimension at least two, $X - Y$ is connected by Proposition 2.4. Since C is algebraically closed we have the same property for U.

This entails V is irreducible. For let $\varphi: C^{2n} \to C^{2n}$ be the isomorphism

$$\varphi(x_1, y_1, \ldots, x_n, y_n) = (x_1 + iy_1, \ldots, x_n + iy_n, x_1 - iy_1, \ldots, x_n - iy_n).$$

Let $V_1, \ldots, V_l$ denote the components of V and let $U_1, \ldots, U_l$ denote the corresponding components of U. Assume that $l > 1$. Then as in (2.1) we have

$$\varphi(U_j) = V_j \times \overline{V}_j \quad \text{and} \quad \varphi(U) = V_1 \times \overline{V}_1 \cup \cdots \cup V_l \times \overline{V}_l.$$

Let $W = \bigcup_{i<j} U_i \cap U_j$. Then $\operatorname{codim}(W, U) \geqslant 2$ since

$$\varphi(W) = \bigcup_{i<j} V_i \cap V_j \times \overline{V_i \cap V_j}$$

has codimension at least two. But the components $U_1 - W, \ldots, U_l - W$ of $U - W$ are disjoint, a contradiction.

Hence V is irreducible and V' is an s.c.i. by Proposition 2.1.

Remark 2.6. The statement and proof of Theorem 2.5 carry over verbatim to the setting of germs of complex analytic varieties and the associated real analytic germs. Instead of our Proposition 2.1, one has Ephraim's result (cf. [**4**, Theorem 2.1]). If ${}_{2n}A$ denotes the ring of germs of real-analytic functions in $2n$ variables then the germ of a real analytic variety is defined by a reduced real ideal. So one again has the bijective correspondence between components of the real analytic germ and the corresponding scheme and Hartshorne's results can again be invoked.

3. Examples. We conclude by giving some examples that illustrate our results.

(3.1) Let $V \subset C^2$ be the union of the z_1 and z_2 axes so that $I_C(V) = (z_1 z_2)$. Then V is connected and is an s.c.i. but is reducible. Let $f = z_1 z_2$.

Then $g = x_1 x_2 - y_1 y_2$ and $h = x_1 y_2 + x_2 y_1$ are the real and imaginary parts of f respectively. So $I_R(V') = \operatorname{rlrad}(g, h) = (x_1, y_1) \cap (x_2, y_2)$.

We note that (g, h) is not a real ideal. For $(x_1 x_2)^2 + (x_1 y_2)^2 = x_1 x_2 g + x_1 y_2 h \in (g, h)$ while $x_1 x_2 \notin (g, h)$.

(3.2) Let $V \subset C^2$ be the union of the lines $z_1 = 0$ and $z_1 = 1$ so that $I_C(V) = (z_1(z_1 - 1))$. Then V is an s.c.i., reducible and disconnected. In this case $V' = V(x_1, y_1) \cup V(x_1 - 1, y_1)$ so that $I_R(V') = (x_1(x_1 - 1), y_1)$ and V' is an s.c.i. Thus without the connectedness assumption the conclusion of Theorem 2.5 is false.

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