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LINEAR CONVOLUTION INTEGRAL EQUATIONS WITH ASYMPTOTICALLY ALMOST PERIODIC SOLUTIONS

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ABSTRACT. Let μ be a bounded Borel measure and f be asymptotically almost periodic. Conditions are found which ensure that certain bounded solutions of the linear convolution integral equation $g * \mu = f$ are asymptotically almost periodic. This result is also extended to the case where the measure μ is replaced by a tempered distribution τ for which convolution with bounded functions makes sense.

1. Classical convolution equations. Recently Fink and Madych [7] studied the asymptotic behavior as $t \to \infty$ of certain bounded solutions of the linear integral equation

$$g * \mu(t) \equiv \int_{-\infty}^{\infty} g(t-s)\, d\mu(s) = f(t), \qquad -\infty < t < \infty, \tag{1}$$

where μ is a bounded Borel measure on $R = (-\infty, \infty)$, and $f(t)$ belongs to $L^\infty = L^\infty(R)$ with $f(t) \to 0$ as $t \to \infty$. In this paper we show that the main result of [7, Theorem 3] may be extended in two directions. First, in this section (see Theorem 1) we show that Theorem 3 of [7] holds for more general forcing functions f than are considered there. Then, in §2 we show that the conclusion of Theorem 1 holds if the measure μ is replaced by a tempered distribution τ for which convolution with bounded functions makes sense.

We will use the notation of [7]. In particular, a function g in L^∞ is said to satisfy the tauberian condition T if

$$\lim_{t\to\infty, s\to 0} |g(t+s) - g(t)| = 0. \tag{T}$$

If m is a positive integer, we say that $f \in L^\infty$ satisfies property $M(f, m)$ if $\lim_{t\to\infty} f(t) = 0$ and $\int_0^\infty t^{m-1}|f(t)|\, dt < \infty$. By $M(f, \infty)$ we mean that $M(f, m)$ holds for each positive integer m. If f is an almost periodic function ($f \in$ a.p.), $\exp f$ denotes the set of exponents of f (see [6, Chapter 3]).

Also, $\mathfrak{M}$ denotes the space of bounded Borel measures. If $\mu \in \mathfrak{M}$, let $\hat{\mu}(\xi) = \int_{-\infty}^{\infty} e^{-i\xi t}\, d\mu(t)$ be the Fourier transform of μ and $\Lambda(\mu) = \{\xi \in R: \hat{\mu}(\xi) = 0\}$ be the set of zeros of $\hat{\mu}$. Finally, we say that $\mu \in \mathfrak{M}$ satisfies property $M\Lambda(\mu)$ if for every $\xi_j \in \Lambda(\mu)$ there exists a positive integer m_j (necessarily unique) such that the function $\hat{\nu}_j$ defined by $\hat{\nu}_j(\xi) = (\xi - \xi_j)^{-m_j}\hat{\mu}(\xi)$ is the Fourier transform of a measure

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$\nu_j \in \mathfrak{M}$ with $\hat{\nu}_j(\xi_j) \neq 0$. Note that if $\Lambda(\mu) = \varnothing$, then μ trivially has property $M\Lambda(\mu)$.

THEOREM 1. *Suppose g is a bounded solution of equation* (1), *where*

(i) $\Lambda(\mu)$ *has no finite accumulation point,*

(ii) μ *satisfies property* $M\Lambda(\mu)$,

(iii) $f = f_1 + f_2$ *where* $f_1 \in$ *a.p. and* f_2 *satisfies property* $M(f, m)$ *for* $m = \sup\{m_j\}$ *where the numbers* m_j *are defined in the previous paragraph,*

(iv) g *satisfies condition* T.

Then $g = g_1 + g_2$ *where* $g_1 \in$ *a.p. with* $\exp g_1 \subseteq \Lambda(\mu) \cup \exp f_1$, *and* $\lim_{t\to\infty} g_2(t) = 0$.

In the case where $f_1(t) \equiv 0$, Theorem 1 is Theorem 3 of [**7**]. (More restricted versions of [**7**, Theorem 3] where, in particular, $\Lambda(\mu)$ is finite are obtained in Jordan and Wheeler [**8**, Theorem 1], and in Levin and Shea's three part paper [**9**, Theorem 5c].) On the other hand, in the case where $f = f_1$ is a.p., a related result which guarantees that bounded, uniformly continuous solutions of (1) (with μ absolutely continuous) are a.p. was obtained by Lewitan [**10**]. (See, also, Doss [**4**] and Lemma 2 below.) Finally, the nature of bounded, uniformly continuous solutions of the homogeneous form of equation (1), i.e., with $f(t) \equiv 0$, was investigated by Beurling in [**2**], [**1**], and now constitutes one aspect of spectral analysis; see [**9**, §8] for a brief discussion which includes references to appropriate literature.

Our proof of Theorem 1 consists of reducing the problem to considering separately the case where $f(t) = f_2(t) \to 0$ as $t \to \infty$, and the case where $f = f_1$ is a.p.

LEMMA 1. *Suppose g is a bounded solution of equation* (1) *which satisfies the tauberian condition T, and $f = f_1 + f_2$ where $f_1 \in$ a.p. and* $\lim_{t\to\infty} f_2(t) = 0$. *Then there exists a function h which is bounded, uniformly continuous on R, and which satisfies $h * \mu(t) = f_1(t)$ for $t \in R$.*

The idea of Lemma 1 is implicitly contained in the discussion given in §20, pp. 567–568 of [**9**]. For completeness we include the proof.

PROOF. The ε-translation set of f_1 is

$$T(f_1, \varepsilon) = \{s \in R\colon |f_1(t+s) - f_1(t)| < \varepsilon \text{ for all } t \in R\}.$$

Since $f_1 \in$ a.p., we can find $s_j \in T(f_1, 1/j)$ for $j = 1, 2, \ldots,$ so that $\lim_{j\to\infty} s_j = \infty$. Since g is bounded and satisfies (T), Lemma 3.2 of [**9**] guarantees that there exist a subsequence $\{s_{j_k}\}$ of $\{s_j\}$ and a bounded, uniformly continuous function h on R such that

$$\lim_{k\to\infty} \left\{ \sup_{|t|<d} |g(t + s_{j_k}) - h(t)| \right\} = 0 \quad \text{for every } d > 0. \tag{2}$$

Fix $t \in R$ and observe that by (1)

$$g * \mu(t + s_{j_k}) = f_1(t + s_{j_k}) + f_2(t + s_{j_k}) \tag{3}$$

for $k = 1, 2, \ldots$. Thus, if we let $k \to \infty$ in (3) and use $\mu \in \mathfrak{M}$, (2), $s_{j_k} \in T(f_1, 1/j_k)$ and $f_2(t) \to 0$ as $t \to \infty$, we get that $h * \mu(t) = f_1(t)$. This completes the proof of Lemma 1.

The following lemma relates properties of bounded, uniformly continuous solutions of equation (1) with $f \in$ a.p. to those of bounded, uniformly continuous solutions of the homogeneous equation

$$g * \mu(t) = 0, \qquad t \in R. \tag{4}$$

LEMMA 2. *Assume that $\mu \in \mathfrak{M}$. If every bounded, uniformly continuous solution g of the homogeneous equation* (4) *is a.p., then, for each $f \in$ a.p., every bounded, uniformly continuous solution g of equation* (1) *is a.p.*

When $\mu(t) \equiv \int_{-\infty}^{t} k(s)\, ds$ with $k \in L^1(R)$, Lemma 2 is due to Doss [**4**, Lemma 2]. The proof for general $\mu \in \mathfrak{M}$ follows in exactly the same manner as is indicated by Doss in [**4**] for the case where μ is absolutely continuous. Namely, use Bochner's proof of Theorem 4 in [**3**] with the differential operator Λg replaced by the integral operator $Kg = g * \mu$ throughout.

PROOF OF THEOREM 1. By Lemma 1 there exists a bounded, uniformly continuous solution h of $h * \mu = f_1$ on R. Since $\Lambda(\mu)$ has no finite accumulation point, every bounded, uniformly continuous solution h_1 of $h_1 * \mu = 0$ is a.p. (See, e.g., [**9**, Proposition 8.1].) Thus, by Lemma 2, $h \in$ a.p. It is easy to verify that $\exp h \subseteq \Lambda(\mu) \cup \exp f_1$.

Now, define $G(t) \equiv g(t) - h(t)$, $t \in R$. Then G satisfies $G * \mu = f_2$ on R, and by Theorem 3 of [**7**], $G = G_1 + G_2$, where $G_1 \in$ a.p. with $\exp G_1 \subseteq \Lambda(\mu)$ and $\lim_{t\to\infty} G_2(t) = 0$. Thus, $g = g_1 + g_2$ where $g_1 \equiv G_1 - h$ and $g_2 \equiv G_2$ satisfy the conclusion of the theorem.

2. Extension to distributions. In this section we show that Theorem 1 holds if the measure μ is replaced by a tempered distribution τ for which, roughly, $g * \tau$ makes sense for bounded g.

We say that a tempered distribution τ on R satisfies property H if the convolution $\tau * \phi \in L^1(R)$ for every ϕ in the Schwartz space $\mathcal{S} = \mathcal{S}(R)$. (The definitions of $\mathcal{S}$, tempered distributions, the convolution $\tau * \phi$, and Fourier transforms of tempered distributions are all standard; e.g., see [**5**].) If $g \in L^\infty$, define the linear form $g * \tau$: $\mathcal{S} \to \mathbf{C}$ by the formula $g * \tau(\phi) = g * (\tau * \tilde{\phi})(0)$ where $\tilde{\phi}(t) \equiv \phi(-t)$ and the first convolution on the right-hand side is taken in the classical sense.

LEMMA 3. *If τ satisfies property H and $g \in L^\infty$, then $g * \tau$ is a tempered distribution.*

PROOF. Since $|g * \tau(\phi)| \leqslant \|g\|_\infty \|\tau * \phi\|_1$, the lemma is true if the mapping $\phi \to \tau * \phi$ is continuous from $\mathcal{S}$ into L^1.

Let ψ be a function in $\mathcal{S}$ whose Fourier transform, $\hat{\psi}$, has the following properties: $\hat{\psi}$ is nonnegative, $\hat{\psi}(\xi) = 1$ if $|\xi| \leqslant 1$, and $\hat{\psi}(\xi) = 0$ if $|\xi| \geqslant 2$. For any positive number r define ψ_r by the formula $\psi_r(t) = r\psi(rt)$. Now suppose that $\{\phi_n\}$ is a sequence in $\mathcal{S}$, $\phi_n \to \phi$ in $\mathcal{S}$, and $\tau * \phi_n \to f$ in L^1. Clearly $\tau * \phi_n * \psi_r$ converges to

$\tau * \phi * \psi_r = f * \psi_r$ in L^1 for each positive r. Now $\|\tau * \phi - f\|_1 < \|\tau * \phi - \tau * \phi * \psi_r\|_1 + \|\tau * \phi * \psi_r - f\|_1$, and choosing r sufficiently large it is clear that the right-hand side of the above inequality can be made arbitrarily small. Hence $\tau * \phi = f$. It follows that the mapping $\phi \to \tau * \phi$ has a closed graph in $\mathbb{S} \times L^1$ and hence must be continuous. (For the variant of the closed graph theorem used here see, for example, [5].)

Various properties of tempered distributions satisfying property H should be clear from the definition. For example, the Fourier transform of such a distribution must be a continuous function.

Clearly, bounded measures and certain linear combinations of their derivatives are tempered distributions which satisfy property H. For example, consider the distribution whose Fourier transform $\hat{\tau}$ is given by

$$\hat{\tau}(\xi) = \sum_{j=1}^{n} P_j(i\xi)\hat{\mu}_j(\xi) \tag{2.1}$$

where the $\hat{\mu}_j$'s are Fourier transforms of bounded measures $\mu_j \in \mathfrak{M}$, and each

$$P_j(i\xi) = \sum_{l=0}^{m_j} a_{j,l}(i\xi)^l, \qquad j = 1, \ldots, n,$$

is a polynomial in $(i\xi)$. Then

$$g * \tau(t) = \sum_{j=1}^{n} \int_{-\infty}^{\infty} P_j(D)g(t-s)\, d\mu_j(s)$$

where $P_j(D)g(t) = \sum_{l=0}^{m_j} a_{j,l}g^{(l)}(t)$ and $g^{(l)}(t)$ is the lth derivative of g.

If τ enjoys property H, the definitions of $\Lambda(\tau)$ and property $M\Lambda(\tau)$ are analogous to those in §1 in the case where $\tau \in \mathfrak{M}$. Namely, $\Lambda(\tau) = \{\xi \in R: \hat{\tau}(\xi) = 0\}$, and we say that τ satisfies property $M\Lambda(\tau)$ if for every $\xi_j \in \Lambda(\tau)$ there exists a positive integer m_j such that the function $\hat{\nu}_j(\xi) \equiv (\xi - \xi_j)^{-m_j}\hat{\tau}(\xi)$ is the Fourier transform of a tempered distribution having property H and $\hat{\nu}_j(\xi_j) \neq 0$.

THEOREM 2. *Suppose τ satisfies property H, g is a bounded solution of $g * \tau = f$ and the following are true*:

(i) $\Lambda(\tau)$ *has no finite accumulation point*,

(ii) τ *satisfies property* $M\Lambda(\tau)$,

(iii) $f = f_1 + f_2$ *where* $f_1 \in$ *a.p. and* f_2 *satisfies property* $M(f, m)$ *for* $m = \sup\{m_j\}$ *where the numbers* m_j *are defined in the previous paragraph*,

(iv) g *satisfies condition* T.

Then $g = g_1 + g_2$ *where* $g_1 \in$ *a.p. with* $\exp g_1 \subseteq \Lambda(\tau) \cup \exp f_1$, *and* $\lim_{t\to\infty} g_2(t) = 0$.

PROOF. Let $\phi(x) = e^{-x^2}$. Clearly, $g * (\tau * \phi) = f * \phi + f_2 * \phi$. It follows from $(\tau * \phi)\hat{}(\xi) = \hat{\tau}(\xi)\hat{\phi}(\xi)$ and $\Lambda(\phi) = \varnothing$, that $\Lambda(\tau * \phi) = \Lambda(\tau)$. Also, for each $\varepsilon > 0$, $T(f_1 * \phi, \varepsilon) \supseteq T(f_1, \varepsilon/\|\phi\|_1)$, so that $f_1 * \phi \in$ a.p.; in addition, by Lemma 4.7 of [6] $\exp(f_1 * \phi) = \exp f_1$. Finally, Lemma 3 of [7] implies that $f_2 * \phi$ satisfies property $M(f_2 * \phi, m)$. Thus, Theorem 2 follows from Theorem 1 with $\mu = \tau * \phi$.

For an example, consider the integrodifferential equation

$$\sum_{l=0}^{n} \int_{-\infty}^{\infty} g^{(l)}(t-s)\, d\mu_l(s) = f(t), \qquad t \in R, \tag{2.2}$$

where $\mu_l \in \mathfrak{M}$, $l = 0, \ldots, n$. Clearly, this equation is a special case of $g * \tau = f$ where τ has property H and $\hat{\tau}(\xi) = \sum_{l=0}^{n} (i\xi)^l \hat{\mu}_l(\xi)$. We say that $\xi_0 \in R$ is a zero of $\hat{\tau}(\xi)$ of multiplicity p if

$$\int_{-\infty}^{\infty} |t|^p |d\mu_l(t)| < \infty \qquad (l = 0, \ldots, n)$$

and if

$$(d^j/d\xi^j)\hat{\tau}(\xi) = 0 \qquad (\xi = \xi_0;\, j = 0, \ldots, p-1)$$

but

$$(d^p/d\xi^p)\hat{\tau}(\xi) \neq 0 \qquad (\xi = \xi_0).$$

Assume that each zero of $\hat{\tau}(\xi)$ has finite multiplicity and let m ($\leqslant \infty$) be the supremum of the multiplicities of the zeros of $\hat{\tau}(\xi)$. Then an argument similar to the proof of Lemma 5 of [7] shows that property $M\Lambda(\tau)$ holds. Thus, if $\Lambda(\tau)$ has no finite accumulation point, if f is as in Theorem 2, and if $g(t)$ is a solution of (2.2) a.e. on R with $g^{(l)} \in L^\infty$ ($l = 0, \ldots, n-1$) and $g^{(n-1)}$ locally absolutely continuous, then g has the form described in Theorem 2. We remark that the special case of equation (2.2) with μ_n the point-mass measure concentrated at $t = 0$, has been considered previously in [**8**, Theorem 5] (see also [**9**, Theorem 5a]) when $\Lambda(\tau)$ is finite.

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