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ON BOUNDEDNESS OF INTEGRABLE AUTOMORPHIC FORMS IN $\mathbf{C}^n$

SU-SHING CHEN

ABSTRACT. We give a necessary and sufficient condition for an integrable automorphic form on a bounded symmetric domain D in $\mathbf{C}^n$ to be bounded.

The question on the boundedness of integrable automorphic forms on the unit disk in $\mathbf{C}^1$ has been investigated in [1], [5], [6], [7], [8], [9] and many other papers. Integrable automorphic forms on the bounded homogeneous domains in $\mathbf{C}^n$ have been considered by Earle [2] and Selberg [10]. In this paper, we shall study the boundedness of integrable automorphic forms on the bounded symmetric domains in $\mathbf{C}^n$.

Let D be a bounded symmetric domain in $\mathbf{C}^n$ with Bergmen kernel function $k(z, w)$, where z and w represent n-tuples $(z_1, \ldots, z_n)$ and $(w_1, \ldots, w_n)$ respectively.

For every holomorphic automorphism g of $\operatorname{Aut}(D)$, we have $k(z, w) = k(gz, gw)g'(z)\overline{g'(w)}$, where $g'(z)$ is the complex Jacobian of the automorphism g. The volume element $dm(z) = k(z, z)dz$ is invariant under the group $\operatorname{Aut}(D)$ of all holomorphic automorphisms of D, where dz is the euclidean volume element of D.

Let Γ be a discrete subgroup of $\operatorname{Aut}(D)$. We choose a fundamental domain R for Γ so that $\partial R \cap D$ has zero volume. A function f holomorphic on D is said to be an automorphic form of dimension $-2q$ if $f(\gamma z)\gamma'(z)^q = f(z)$ for all z in D and γ in Γ. We denote by $A_q(\Gamma)$ the space of integrable forms, i.e., the set of all holomorphic automorphic forms f of dimension $-2q$ such that

$$\|f\|_q = \int_R |f(z)|\, |k(z, z)|^{-q/2} dm(z) < \infty.$$

We denote by $B_q(\Gamma)$ the space of bounded forms, i.e., the set of all forms of dimension $-2q$ such that

$$\|f\|_\infty = \sup_{z \in D} |f(z)k(z, z)^{-q/2}| < \infty.$$

We refer to the paper [2] of Earle for notations and basic facts. In particular, q is any integer $\geqslant 2$ so that all formulas in [2] are valid. $c(q)$ is a certain constant depending only on q.

The following theorem is a generalization of a result of Metzger and Rao [7].

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THEOREM. *Let D be a bounded symmetric domain in $\mathbf{C}^n$ and let Γ be a discrete subgroup of* $\operatorname{Aut}(D)$. *For $q \geqslant 2$, $A_q(\Gamma) \subset B_q(\Gamma)$ if and only if*

$$\sup_{z\in D} |k(z, z)^{-q}\alpha(z, z)| < \infty, \tag{**}$$

where $\alpha(z, w) = c(q)\Sigma_{\gamma\in\Gamma}k(\gamma z, w)^q\gamma'(z)^q$.

PROOF. According to Theorem 3.1, Corollary 5.2 and Theorem 7 of [2], the function $\alpha(z, w)$, for a fixed w in D, belongs to $A_q(\Gamma)$ and $B_q(\Gamma)$.

If f is in $A_q(\Gamma)$, then by [2], we have

$$\begin{aligned} f(w) &= c(q)\int_D f(z)k(w, z)^q k(z, z)^{-q}\, dm(z) \\ &= c(q)\int_R \sum_{\gamma\in\Gamma} f(\gamma z)k(w, \gamma z)^q k(\gamma z, \gamma z)^{-q}\, dm(z) \\ &= c(q)\int_R \sum_{\gamma\in\Gamma} f(\gamma z)k(w, \gamma z)^q k(z, z)^{-q}\gamma'(z)^q\, \overline{\gamma'(z)}^{\,q}\, dm(z) \\ &= c(q)\int_R \sum_{\gamma\in\Gamma} k(w, \gamma z)^q\, \overline{\gamma'(z)}^{\,q} f(z)k(z, z)^{-q}\, dm(z) \\ &= c(q)\int_R f(z)\,\overline{\alpha(z, w)}\, k(z, z)^{-q}\, dm(z). \end{aligned}$$

Consequently,

$$|f(w)|\,|k(w, w)^{-q/2}| \leqslant c(q)\|f\|_q \sup_{z,w\in D} |\alpha(z, w)|\,|k(w, w)|^{-q/2}|k(z, z)|^{-q/2}.$$

Since $\alpha(z, w)$ is in $A_q(\Gamma)$ for a fixed w in D,

$$\alpha(w, w) = c(q)\int_R |\alpha(z, w)|^2 k(z, z)^{-q}\, dm(z)$$

and

$$\alpha(z, w) = c(q)\int_R \alpha(w', w)\,\overline{\alpha(w', z)}\, k(w', w')^{-q}\, dm(w').$$

The Schwarz inequality implies that $|\alpha(z, w)|^2 \leqslant |\alpha(z, z)\alpha(w, w)|$. Thus f is in $B_q(\Gamma)$.

Conversely, by the closed graph theorem, we have a bounded linear map from $A_q(\Gamma)$ into $B_q(\Gamma)$. Thus, there exists a positive constant C such that for all f in $A_q(\Gamma)$ and z in D

$$|f(z)|\,|k(z, z)^{-q/2}| \leqslant C\|f\|_q.$$

For the function $\alpha(\cdot, w)$, and z, in D,

$$|\alpha(z, w)|\,|k(z, z)^{-q/2}| \leqslant C\|\alpha(\cdot, w)\|_q$$

and

$$|\alpha(w, w)|\,|k(w, w)^{-q/2}| \leqslant C\|\alpha(\cdot, w)\|_q.$$

But

$$\|\alpha(\cdot, w)\|_q \leqslant c(q)\int_R \sum_{\gamma\in\Gamma} |k(\gamma z, w)|^q |\gamma'(z)|^q |k(z, z)^{-q/2}| \, dm(z)$$

$$= c(q)\int_D |k(z, w)^q| \, |k(z, z)^{-q/2}| \, dm(z)$$

$$= c(q)/c(q/2)|k(w, w)^{q/2}|$$

by a formula in [2]. Thus (**) is satisfied.

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Department of Mathematics, University of Florida, Gainesville, Florida 32611