PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 78, Number 3, March 1980

LIFTINGS OF FUNCTIONS WITH VALUES IN A COMPLETELY REGULAR SPACE

G. A. EDGAR[1] AND MICHEL TALAGRAND

ABSTRACT. Let T be a completely regular space, let $(\Omega, \mathfrak{F}, \mu)$ be complete probability space, and let ρ: $\mathcal{L}^\infty(\mu) \to \mathcal{L}^\infty(\mu)$ be a lifting. If f: $\Omega \to T$ is a Baire measurable function, must there exist a function $\tilde{f}$ with almost all of its values in T, such that $\rho(h \circ f) = h \circ \tilde{f}$ for all bounded continuous functions h on T? If T is strongly measure-compact, then the answer is "yes". If T is not measure-compact, then the answer is "no". This shows that a lifting is not always the best method for the construction of weak densities for vector measures.

Let T be a completely regular Hausdorff space, and let $C_b(T)$ be the set of bounded continuous real-valued functions on T. Thus T is homeomorphically embedded in $\mathbf{R}^{C_b(T)}$ (in the product topology) by the map Φ: $T \to \mathbf{R}^{C_b(T)}$ defined by $\Phi(t)(h) = h(t)$, $t \in T$, $h \in C_b(T)$. We will identify T with its image $\Phi[T]$. Any bounded continuous function on T extends to a bounded continuous function on $\mathbf{R}^{C_b(T)}$, so the Baire sets in T are the intersections of T with the Baire subsets of $\mathbf{R}^{C_b(T)}$.

Let $(\Omega, \mathfrak{F}, \mu)$ be a complete probability space. A function f: $\Omega \to T$ is Baire measurable if and only if $h \circ f$ is measurable for all $h \in C_b(T)$. Let ρ be a lifting for $\mathcal{L}^\infty(\Omega, \mathfrak{F}, \mu)$ (see [3, p. 34] for the definition). Define $\rho'(f)$: $\Omega \to \mathbf{R}^{C_b(T)}$ by $\rho'(f)(\omega)(h) = \rho(h \circ f)(\omega)$, $\omega \in \Omega$, $h \in C_b(T)$. Then $\rho'(f)$ is Baire measurable (in fact Borel measurable [3, p. 52]), and for any $h \in C_b(\mathbf{R}^{C_b(T)})$, we have $h \circ \rho'(f) = h \circ f$ a.e. and $\rho(h \circ f) = h \circ \rho'(f)$ everywhere [3, §IV. 5].

Under what circumstances does the lifting $\rho'(f)$ of f have its values in $T \subseteq \mathbf{R}^{C_b(T)}$? A well-known sufficient condition is that $f[\Omega]$ be relatively compact in T [3, p. 52]. In fact, for nonatomic μ, if ρ varies over all liftings for $\mathcal{L}^\infty(\mu)$, then $\rho'(f)(\omega)$ varies over all values in the essential range $\{y \in \mathbf{R}^{C_b(T)}\colon \mu(f^{-1}(U)) > 0$ for all neighborhoods U of $y\}$, which is a compact subset of $\mathbf{R}^{C_b(T)}$. Thus, in order that $\rho'(f)[\Omega] \subseteq T$ for all liftings ρ, it is necessary and sufficient that the essential range of f be compact in T. This can be seen as follows. Let y_0 belong to the essential range of f, and choose $\omega_0 \in \Omega$ with $\mu(\{\omega_0\}) = 0$. Let $\mathfrak{U}$ be a maximal filter of sets from $\mathfrak{F}$ of positive measure that includes $\{f^{-1}(U)\colon U$ is an open neighborhood of $y_0\}$. Then any lifting ρ for $\mathcal{L}^\infty(\mu)$ remains a lifting when redefined at ω_0 as the limit along $\mathfrak{U}$:

$$\rho(g)(\omega_0) = \lim_{\omega, \mathfrak{U}} g(\omega), \qquad g \in \mathcal{L}^\infty(\mu).$$

Then $\rho'(f)(\omega_0) = y_0$.

Received by the editors August 23, 1978 and, in revised form, April 26, 1979.
AMS (MOS) subject classifications (1970). Primary 46G15.
[1]Supported in part by National Science Foundation grant number MCS77-04049.

In this paper we are interested in a related question: Under what circumstances does $\rho'(f)$ have almost all of its values in T?

We will be considering three classes of measures on a completely regular space T. First $P_\sigma(T)$ denotes the set of probability measures on the σ-algebra of Baire sets in T. Second, $P_\tau(T)$ denotes the set of *τ-smooth* probability measures, i.e. measures $\lambda \in P_\sigma(T)$ such that $\int h_\alpha \, d\lambda \to 0$ for any net $h_\alpha \in C_b(T)$ which decreases pointwise to 0. Finally, $P_t(T)$ denotes the set of *tight* probability measures, i.e. measures $\lambda \in P_\sigma(T)$ such that, for every $\varepsilon > 0$, there is a compact $K \subseteq T$ with $\inf\{\lambda(B)\colon B \supseteq K, B \text{ is a Baire set}\} \geqslant 1 - \varepsilon$. (These are the measures which extend to Radon measures on T.)

A completely regular space T is called *universally Radon measurable* iff $P_\tau(T) = P_t(T)$ (justification for this terminology can be found in **[8]** or in **[4]**). A space T is called *measure-compact* iff $P_\sigma(T) = P_\tau(T)$ **[5]**. A space T is called *strongly measure-compact* iff $P_\sigma(T) = P_t(T)$ **[5]**, i.e. T is both universally Radon measurable and measure-compact. Examples of spaces T which do and do not have these properties can be found in **[5]**.

THEOREM 1. *Let T be a completely regular Hausdorff space. Then* (a′) $\Rightarrow$ (b) $\Rightarrow$ (a″).

(a′) *T is strongly measure-compact.*

(b) *If $(\Omega, \mathfrak{F}, \mu)$ is a complete probability space, if ρ is a lifting for $\mathfrak{L}^\infty(\mu)$, and if $f\colon \Omega \to T$ is Baire measurable, then $\rho'(f)(\omega) \in T$ for almost all $\omega \in \Omega$.*

(a″) *T is measure-compact.*

If T is universally Radon measurable, then the three conditions are equivalent.

With the same proof (but twice as many definitions) we can obtain a more general result.

Let T be a completely regular space, and let $H \subseteq C_b(T)$ be a set of functions which determines the topology in T, i.e. $\mathfrak{R}_H = \{h^{-1}(U)\colon h \in H, U \subseteq \mathbf{R} \text{ is open}\}$ is a subbase for the topology. We will assume H is a uniformly closed algebra which contains the constants. (For example, if $T \subseteq S$ where S is completely regular, let H be the set of functions in $C_b(T)$ which have continuous extensions to S.) Then $\Phi\colon T \to \mathbf{R}^H$ defined by $\Phi(t)(h) = h(t)$ identifies T homeomorphically with a subset of $\mathbf{R}^H$. Let $\mathfrak{B}_H$ be the σ-algebra generated by $\mathfrak{R}_H$, so that $\mathfrak{B}_H$ consists of the sets $T \cap B$, where B is a Baire set in $\mathbf{R}^H$. In general, not every bounded continuous function on T can be extended to $\mathbf{R}^H$, and $\mathfrak{B}_H$ may be strictly smaller than the σ-algebra of Baire sets of T.

Let $(\Omega, \mathfrak{F}, \mu)$ be a complete probability space, and let ρ be a lifting for $\mathfrak{L}^\infty(\mu)$. If $f\colon \Omega \to T$ is $\mathfrak{B}_H$-measurable, then $\rho'(f)(\omega)(h) = \rho(h \circ f)(\omega)$, $\omega \in \Omega$, $h \in H$, defines $\rho'(f)\colon \Omega \to \mathbf{R}^H$.

Let λ be a probability measure on $\mathfrak{B}_H$. Then λ is called *tight* iff, for every $\varepsilon > 0$, there is a compact set $K \subseteq T$ such that $\inf\{\lambda(B)\colon B \in \mathfrak{B}_H, B \supseteq K\} \geqslant 1 - \varepsilon$; or, equivalently, iff λ extends to a Radon measure on T. Also, λ is called *τ-smooth* iff for any net $f_\alpha \in H$ which decreases to 0, we have $\lim \int f_\alpha \, d\lambda = 0$. (Since H generates the topology of T, a measure λ on $\mathfrak{B}_H$ is τ-smooth if and only if it extends to a τ-smooth measure on the Baire sets of T.)

THEOREM 2. *Let T, H, $\mathfrak{B}_H$ be as above. Then* (a′) ⇒ (b) ⇒ (a″).

(a′) *Every probability measure on $\mathfrak{B}_H$ is tight.*

(b) *If $(\Omega, \mathfrak{F}, \mu)$ is a complete probability space, if ρ is a lifting for $\mathfrak{L}^\infty(\mu)$, and if f: $\Omega \to T$ is $\mathfrak{B}_H$-measurable, then $\rho'(f)(\omega) \in T$ for almost all $\omega \in \Omega$.*

(a″) *Every probability measure on $\mathfrak{B}_H$ is τ-smooth.*

If T is universally Radon measurable, then the three conditions are equivalent.

PROOF. (a′) ⇒ (b). Let $(\Omega, \mathfrak{F}, \mu)$, ρ, and f be given. Let $\lambda = f(\mu)$ be the image measure, defined by $\lambda(B) = \mu(f^{-1}(B))$ for $B \in \mathfrak{B}_H$, or by $\int h\, d\lambda = \int h \circ f\, d\mu$ for $h \in H$. By the assumption (a′), λ is tight. Let $\varepsilon > 0$ be given. There is a compact set $K \subseteq T$ so that $\lambda(B) \geqslant 1 - \varepsilon$ for every $B \in \mathfrak{B}_H$ with $B \supseteq K$. Then

$$C = \bigcap \left\{\rho(A) \colon A = f^{-1}(B), B \in \mathfrak{B}_H, B \supseteq K\right\}$$

is in $\mathfrak{F}$, $\mu(C) \geqslant 1 - \varepsilon$, and $\rho(C) \subseteq C$ [3, p. 40]. We claim that $\rho'(f)(\omega) \in K$ for all $\omega \in \rho(C)$. Indeed, suppose $y \in \mathbf{R}^H \setminus K$. Then there is a continuous function g: $\mathbf{R}^H \to [0, 1]$ with $g(y) = 1$ and $g(t) = 0$ for all $t \in K$. Now ([3, p. 52]; $f[\Omega]$ is relatively compact in $\mathbf{R}^H$) since $g \circ f = 0$ a.e. on C, $g \circ \rho'(f) = 0$ everywhere on $\rho(C)$. So, if $\omega \in \rho(C)$, we have $\rho'(f)(\omega) \neq y$. This shows $\rho'(f)[\rho(C)] \subseteq K \subseteq T$. Now $\mu(\rho(C)) \geqslant 1 - \varepsilon$, and ε was chosen arbitrarily, so $\rho'(f)(\omega) \in T$ for almost all $\omega \in \Omega$.

(b) ⇒ (a″). Let λ be a probability measure on $\mathfrak{B}_H$. Let $\Omega = T$, let $(\Omega, \mathfrak{F}, \mu)$ be the completion of $(T, \mathfrak{B}_H, \lambda)$ and let f: $\Omega \to T$ be the identity function. Suppose ρ is a lifting for $\mathfrak{L}^\infty(\mu)$. By hypothesis (b), $\rho'(f)(\omega) \in T$ for almost all $\omega \in \Omega$. Let $h_\alpha \in H$ decrease pointwise to 0 on T. Let $h'_\alpha \in C_b(\mathbf{R}^H)$ be an extension of h_α. Then $h'_\alpha \circ \rho'(f)$ decreases to 0 a.e., and $\rho(h_\alpha \circ f) = h'_\alpha \circ \rho'(f)$, so $h'_\alpha \circ \rho'(f)$ decreases everywhere and

$$\lim \int h'_\alpha \circ \rho'(f)\, d\mu = \int \lim h'_\alpha \circ \rho'(f)\, d\mu = 0$$

[3, p. 40]. Thus

$$\lim \int h_\alpha\, d\lambda = \lim \int h_\alpha \circ f\, d\lambda$$
$$= \lim \int h_\alpha \circ f\, d\mu = \lim \int \rho(h_\alpha \circ f)\, d\mu = 0.$$

Therefore λ is τ-smooth.

Finally, if T is universally Radon measurable, we have (a″) ⇒ (a′). If λ is a probability measure on $\mathfrak{B}_H$, then by assumption (a″), λ is τ-smooth. But then λ extends to a τ-smooth Baire measure on T, which is tight since T is universally Radon measurable. Hence λ is tight. □

Neither of the implications in Theorem 1 can be reversed in general, as the following two examples show.

EXAMPLE 3. (a″) $\not\Rightarrow$ (b). Write τ for the usual topology on [0, 1], and λ for Lebesgue measure on [0, 1]. By the well-ordering theorem [7, Theorem 5.3], there is a set $M \subseteq [0, 1]$ such that $F \cap M \neq \varnothing$ and $F \setminus M \neq \varnothing$ for all uncountable closed sets F in $[0, 1]$. In particular, M has inner measure 0 and outer measure 1. Let $\Omega = M$, and let μ be the measure induced on Ω by λ, i.e. $\mu(\Omega \cap A) = \lambda(A)$ for any Lebesgue measurable set A.

The topological space T will be $[0, 1]$ together with the toplogy τ' whose open sets are all sets of the form $G \cup P$, where G is τ-open and P is any subset of M. Equivalently, τ' is the topology determined by the set $\{\chi_{\{s\}}: s \in M\} \cup \{i\}$ of real-valued functions on T, where i is the identity function. A set $A \subseteq T$ is τ'-closed if and only if it can be written in the form $A = F \setminus P$, where F is τ-closed and $P \subseteq M$. In particular, if A is τ'-closed and $A \subseteq M$, then $A \subseteq F \subseteq M$ for some τ-closed set F which (being disjoint from $[0, 1] \setminus M$) must be countable.

Now consider a set A which is a τ'-(open F_σ)-set. Since A is τ'-open, $A = G \cup P$, where G is τ-open and $P \subseteq M \setminus G$. Also, $A = \bigcup_{n=1}^{\infty} A_n$, where A_n is τ'-closed, so $A_n \cap P = A_n \setminus G$ is τ'-closed and contained in M, hence countable. Then $P = \bigcup_{n=1}^{\infty}(P \cap A_n)$ is countable.

Define f: $\Omega \to T$ by $f(\omega) = \omega$. If A is a τ'-(open F_σ)-set, then $A = G \cup P$, where G is τ-open and P is countable, so A is Lebesgue measurable, and hence $f^{-1}(A) = \Omega \cap A$ is μ-measurable. This shows that f is a Baire measurable function from Ω to T. Suppose (for purposes of contradiction) that $\rho'(f)(\omega) \in T$ for almost all $\omega \in \Omega$. Now for $\omega \in \Omega$, the function $\chi_{\{\omega\}}$ is a continuous function on T, and $\chi_{\{\omega\}} \circ f = 0$ a.e., so $\chi_{\{\omega\}} \circ \rho'(f) = 0$ everywhere, so $\rho'(f)(\omega) \neq \omega$. On the other hand, the identity function i: $T \to \mathbf{R}$ is continuous on T, so $i \circ f = i \circ \rho'(f)$ a.e., that is, $\rho'(f)(\omega) = \omega$ for almost all $\omega \in \Omega$. This contradiction shows that $\rho'(f)(\omega)$ is not in T for almost all ω.

Finally, we claim that T is measure-compact. It suffices to show that T is Lindelof [**9**, p. 175]. Let $\{A_i: i \in I\}$ be a cover of $[0, 1]$ by τ'-open sets. Write $A_i = G_i \cup P_i$, where G_i is τ-open and $P_i \subseteq M$. Let $G = \bigcup_{i \in I} G_i$. There is a countable set $I_1 \subseteq I$ with $G = \bigcup_{i \in I_1} G_i$, since (G, τ) is separable and metrizable. Now $[0, 1] \setminus G = (\bigcup_{i \in I} A_i) \setminus G \subseteq \bigcup_{i \in I} P_i \subseteq M$, and $[0, 1] \setminus G$ is τ-closed, so it is countable. Thus, there is a countable set $I_2 \subseteq I$ such that $[0, 1] \setminus G \subseteq \bigcup_{i \in I_2} P_i$. Thus $\{A_i: i \in I_1 \cup I_2\}$ is a countable subcover of $[0, 1]$. This shows that (T, τ') is Lindelof.

EXAMPLE 4. (b) $\not\Rightarrow$ (a). Let M, μ be as in Example 3. Let T be M with the usual topology (τ). Then μ is a Baire measure on T which is not tight. This shows that T is not strongly measure-compact. But if $(\Omega, \mathfrak{F}, \lambda)$ is any complete probability space, ρ is a lifting for $\mathcal{L}^\infty(\lambda)$, and f: $\Omega \to T$ is Baire measurable, then (since the inclusion map $T \to \mathbf{R}$ is continuous and bounded) $\rho'(f) = f$ a.e., so $\rho'(f)(\omega) \in T$ for almost all $\omega \in \Omega$.

The work in this paper started from the following question of A. Goldman: Is lifting the best way to construct weak densities for vector measures? (See [**2**].) The methods used in Theorem 2 will also prove the following.

THEOREM 5. *Let* $(\Omega, \mathfrak{F}, \mu)$ *be a complete probability space, let E be a locally convex space, let* $\mathbf{m}$: $\mathfrak{F} \to E$ *be a vector measure with bounded* μ*-average range, and let* λ *be the associated cylindrical measure. Then* (a) $\Rightarrow$ (b) $\Rightarrow$ (c) $\Rightarrow$ (d).

(a) λ *is* (*the restriction of*) *a Radon measure.*

(b) *Every lifting gives a density for* $\mathbf{m}$.

(c) *There is a lifting which gives a density for* $\mathbf{m}$.

(d) λ *is* τ*-smooth.*

Our last example illustrates a situation where weak densities exist, but cannot be obtained from liftings.

EXAMPLE 6. Suppose E is a Banach space with the following three properties. (a) Every vector measure with bounded average range in E has a weak density. (b) Every bounded scalarly measurable function with values in E is Pettis integrable. (c) There is a bounded scalarly measurable function f with values in E which is not weakly equivalent to a Bochner measurable function. Then, if $\mathbf{m}$ is the indefinite Pettis integral of f, defined by $\mathbf{m}(A) = \int_A f\,d\mu$, the associated cylindrical measure $\lambda = f(\mu)$ is not τ-smooth [**1**, p. 672], so $\rho'(f)(\omega)$ is not in E for almost all ω. In such a space E, every measure with bounded average range has a weak density, but such densities cannot always be obtained by liftings in the weak topology of E.

Examples of spaces satisfying the properties (a), (b), (c) include a nonseparable dual of a separable Banach space which does not have a subspace isomorphic to l^1, such as the dual of the James Tree space [**6**, Theorem 3].

REFERENCES

1. G. A. Edgar, *Measurability in a Banach space*, Indiana Univ. Math. J. **26** (1977), 663–677.

2. A. Goldman, *Mesures cylindriques, mesures vectorielles et questions de concentration cylindrique*, Pacific J. Math. **69** (1977), 385–413.

3. A. Ionescu Tulcea and C. Ionescu Tulcea, *Topics in the theory of liftings*, Ergebnisse der Mathematik und Ihrer Grenzgebiete, vol. 48, Springer-Verlag, Berlin and New York, 1969.

4. J. D. Knowles, *Measures on topological spaces*, Proc. London Math. Soc. **17** (1967), 139–156.

5. W. Moran, *Measures and mappings on topological spaces*, Proc. London Math. Soc. **19** (1969), 493–508.

6. K. Musial, *The weak Radon-Nikodym property in Banach spaces*, Studia Math. **64** (1979), 151–173.

7. J. C. Oxtoby, *Measure and category*, Graduate Texts in Mathematics, vol. 2, Springer-Verlag, Berlin and New York, 1971.

8. C. Sunyach, *Une caractérisation des espaces universellement Radon-measurables*, C. R. Acad. Sci. Paris Sér. A-B **268** (1969), 864–866.

9. V. S. Varadarajan, *Measures on topological spaces*, Amer. Math. Soc. Transl. (2) **48** (1965), 161–228.

DEPARTMENT OF MATHEMATICS, OHIO STATE UNIVERSITY, COLUMBUS, OHIO 43210

ÉQUIPE D'ANALYSE, UNIVERSITÉ PARIS VI, 75230 PARIS CEDEX 05, FRANCE