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PERIODIC SOLUTIONS FOR A CLASS OF ORDINARY DIFFERENTIAL EQUATIONS

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ABSTRACT. A T-periodic solution to the differential equation $x'' + cx' + g(x) = f(t) \equiv f(t + T)$ is shown to exist whenever a simple condition on g holds, provided $c \neq 0$. No assumption is made concerning the growth of g. The condition on g is necessary if g is either an increasing or a decreasing function.

In this note we discuss existence of T-periodic solutions for the differential equation

$$x'' + cx' + g(x) = f(t) \equiv f(t + T). \tag{1.1}$$

We assume that g: $R \to R$ is continuous, f: $R \to R$ is continuous and T-periodic, and $c \in R$. Here R denotes the real numbers and $T > 0$.

Let $m = T^{-1}\int_0^T f(t)\,dt$. We prove the following theorem.

THEOREM 1. *Suppose the following conditions are satisfied*:

(i) *There is a number* $r \geqslant 0$ *such that* $(g(x) - m)x \geqslant 0$ $(\leqslant 0)$ *whenever* $|x| \geqslant r$;

(ii) $c \neq 0$.

Then there is at least one T-periodic solution of (1.1).

Theorem 1 extends a result due to A. C. Lazer [2]. Lazer showed that under condition (i) there is a T-periodic solution for any $c \in R$ provided g is sublinear; that is, provided $g(x)/x \to 0$ as $|x| \to +\infty$. Thus, we show that if $c \neq 0$ sublinearity is not needed. More recently, under the hypothesis that $c = 0$, Fucik and Lovicar [1] showed that there is a T-periodic solution for any T-periodic forcing function f, provided $g(x)/x \to +\infty$ as $|x| \to +\infty$, thus solving an old problem. Theorem 1 includes an extension of the result of [1] to the case $c \neq 0$. We emphasize that we need no growth condition on g. Our proof is based on a simple Leray-Schauder argument.

In what follows we let $S = L^2(0, T)$ with norm $\|\cdot\|$ given by $\|u\|^2 = \int_0^T u^2\,dt$ for $u \in S$. Let AC be the functions absolutely continuous on $[0, T]$. By H^1 we will mean the space of functions $u \in AC$ with $u' \in S$ $(' \equiv d/dt)$, with norm $|u|_1$ given by

$$|u|_1 = \max_{0 \leqslant t \leqslant T} |u(t)| + \|u'\|.$$

It is well known that H^1 is a Banach space.

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PROOF OF THEOREM 1. There is no loss of generality in supposing $m = 0$, and we will so assume. We consider the case $(g(x) - m)x > 0$ if $|x| > r$. The proof in the other case is essentially the same.

Let $L\colon D(L) \subseteq H^1 \to S$ be defined as follows:

$$D(L) = \{u \in H^1: u' \in AC, u'' \in S, u(0) = u(T), u'(0) = u'(T)\},$$

$Lu = u''$ for $u \in D(L)$.

Choose $b > 0$ with $b \neq 4n^2\pi^2/T^2$ for any integer n. Then $(L + bI)^{-1}$ exists and maps S into H^1 compactly. There is a T-periodic solution x of (1.1) if and only if the restriction of x to $[0, T]$ is a solution to

$$x = (L + bI)^{-1}(f + bx - g(x) - cx'). \tag{1.2}$$

The mapping $x \to Nx \equiv f + bx - g(x) - cx'$ maps H^1 into S continuously, and thus the mapping $x \to (L + bI)^{-1}Nx$ is a completely continuous self map of H^1. By the Leray-Schauder degree theory (see, e.g., Lloyd [3]) there is a solution to the equation $x = (L + bI)^{-1}Nx$ if there is an *a priori* bound in H^1 on the possible solutions to the family of equations

$$x = \lambda(L + bI)^{-1}Nx, \qquad 0 < \lambda < 1. \tag{1.3}$$

It is easily seen that $x \in H^1$ is a solution of (1.3) if and only if $x \in D(L)$ and x extends on R to a T-periodic solution of the equation

$$x'' + (1 - \lambda)bx + \lambda cx' + \lambda g(x) = \lambda f. \tag{1.4}$$

Suppose x is a T-periodic solution of (1.4). Multiplying each side of (1.4) by x', integrating the resulting equation from 0 to T, and using $x(0) = x(T)$, $x'(0) = x'(T)$, we obtain

$$\lambda \int_0^T c|x'|^2\, dt = \lambda \int_0^T fx'\, dt.$$

Thus, for $0 < \lambda < 1$, we have

$$\|x'\| \leqslant \|f\|/|c|. \tag{1.5}$$

Now integrating each side of equation (1.4) from 0 to T we obtain

$$(1 - \lambda)b\int_0^T x(t)\, dt + \lambda\int_0^T g(x(t))\, dt = 0,$$

since $m = 0$. By the mean value theorem for integrals, there is a number z, $0 \leqslant z \leqslant T$, such that

$$(1 - \lambda)bx(z) + \lambda g(x(z)) = 0. \tag{1.6}$$

Since $b > 0$ and $g(x)x \geqslant 0$ if $|x| \geqslant r$, (1.6) implies that $|x(z)| \leqslant r$. We now may write

$$x(t) = x(z) + \int_z^t x'(s)\, ds$$

and thus

$$|x(t)| \leqslant r + T^{1/2}\|x'\| \tag{1.7}$$

by the Cauchy-Schwarz inequality. Combining (1.5) and (1.7) we have, for any solution x and any λ, $0 < \lambda < 1$,

$$|x|_1 \leqslant r + T^{1/2}\|x'\| + \|x'\| \leqslant r + (T^{1/2} + 1)\|f\|/|c|. \tag{1.8}$$

By (1.8) all possible solutions of (1.3) are bounded in H^1, independently of λ, $0 < \lambda < 1$. The existence of a T-periodic solution follows by the Leray-Schauder alternative, as indicated earlier. Q.E.D.

REMARKS. Let $h: R \to R$ be continuous. Then a minor modification in the proof of Theorem 1 can be used to show that the equation

$$x'' + h(x)x' + g(x) = f(t) \equiv f(t + T) \tag{1.9}$$

has a T-periodic solution provided (i) holds and there is a number $c > 0$ such that either $h(x) \geqslant c$ for all $x \in R$ or $h(x) \leqslant -c$ for all $x \in R$.

It is easy to verify that (i) is also necessary if g is an increasing (decreasing) function. Integrate (1.9) from 0 to T, apply the mean value theorem for integrals and then use the assumption that g is increasing (decreasing).

NOTE ADDED IN PROOF. Shortly before proofs for this paper were received a paper of J. Bebernes and M. Martelli, *Periodic solutions for Liénard systems*, appeared in Bol. Un. Mat. Ital. (5) **16-A** (1979), 398–405. They obtain the same results by a different (and slightly longer) method.

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