PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 78, Number 3, March 1980

A NEW CRITERION FOR p-VALENT FUNCTIONS

R. M. GOEL AND N. S. SOHI

ABSTRACT. In this paper we consider the classes K_{n+p-1} of functions $f(z) = z^p + a_{p+1}z^{p+1} + \cdots$ which are regular in the unit disc $E = \{z: |z| < 1\}$ and satisfying the condition

$$\operatorname{Re}\left((z^n f)^{(n+p)} / (z^{n-1} f)^{(n+p-1)}\right) > (n+p)/2,$$

where p is a positive integer and n is any integer greater than $-p$. It is proved that $K_{n+p} \subset K_{n+p-1}$. Since K_0 is the class of p-valent functions, consequently it follows that all functions in K_{n+p-1} are p-valent. We also obtain some special elements of K_{n+p-1} via the Hadamard product.

1. Introduction. Let $A(p)$ denote the class of functions

$$f(z) = z^p + a_{p+1}z^{p+1} + \cdots, \qquad p \text{ a positive integer,} \tag{1}$$

which are regular in the unit disc $E = \{z: |z| < 1\}$. In this paper we shall prove that a function $f \in A(p)$ and satisfying one of the conditions

$$\operatorname{Re}\left(\frac{(z^n f)^{(n+p)}}{(z^{n-1} f)^{(n+p-1)}}\right) > \frac{n+p}{2}, \qquad z \in E, \tag{2}$$

n any integer greater than $-p$, is p-valent in E. We denote by K_{n+p-1} the classes of functions $f(z) \in A(p)$ and satisfying (2). We shall show that

$$K_{n+p} \subset K_{n+p-1}. \tag{3}$$

Since K_0 is the class of functions which satisfy the condition

$$\operatorname{Re} \frac{zf'(z)}{f(z)} > \frac{2p-1}{2} > 0$$

and we know from [4] that such functions are p-valent, the p-valence of functions in K_{n+p-1} will follow from (3). By taking $p = 1$, we get the classes K_n introduced by S. Ruscheweyh in [3] and thus our results are generalizations of Ruscheweyh's.

2. The classes K_{n+p-1}. Let f and g belong to $A(p)$. We denote by $f * g$ the Hadamard product or convolution of $f, g \in A(p)$, that is, if

$$f(z) = z^p + \sum_{n=1}^{\infty} a_{p+n} z^{p+n}, \qquad g(z) = z^p + \sum_{n=1}^{\infty} b_{p+n} z^{p+n},$$

then

$$f(z) * g(z) = z^p + \sum_{n=1}^{\infty} a_{p+n} b_{p+n} z^{p+n}.$$

Received by the editors January 4, 1979 and, in revised form, April 26,1979.
AMS (MOS) subject classifications (1970). Primary 30A32, 30A36.
Key words and phrases. p-valent functions, p-valent starlike functions, Hadamard product.

Let

$$D^{n+p-1}f(z) = \frac{z^p}{(1-z)^{n+p}} * f(z); \tag{4}$$

then

$$D^{n+p-1}f(z) = \frac{z^p\left(z^{n-1}f(z)\right)^{(n+p-1)}}{(n+p-1)!}. \tag{5}$$

From (2) and (5) it follows that a function f in $A(p)$ belongs to K_{n+p-1} if and only if

$$\operatorname{Re}\left(\frac{D^{n+p}f}{D^{n+p-1}f}\right) > \frac{1}{2}, \qquad z \in E. \tag{6}$$

In the notation (6), we can also define a class K_{-1} as the set of functions $f \in A(p)$ with $\operatorname{Re}(f(z)/z^p) > \frac{1}{2}$, $z \in E$.

We shall need the following lemma due to I. S. Jack [1].

LEMMA 1. *Let w be nonconstant and regular in $|z| < R$, $w(0) = 0$. If $|w|$ attains its maximum value on the circle $|z| = r < R$, at z_0, we have $z_0 w'(z_0) = kw(z_0)$ where k is a real number and $k \geqslant 1$.*

THEOREM 1. $K_{n+p} \subset K_{n+p-1}$.

PROOF. Let $f \in K_{n+p}$,

$$g(z) = z^p\left(\frac{D^{n+p-1}f(z)}{z^p}\right)^{2p/(n+p)} \tag{7}$$

and $R = \sup\{r\colon g(z) \neq 0,\ 0 < |z| < r < 1\}$. Then $u(z) = zg'(z)/g(z)$ is regular in this circle. (4) yields the identity

$$z\left(D^{n+p-1}f\right)' = (p+n)D^{n+p}f - nD^{n+p-1}f. \tag{8}$$

Differentiating (7) logarithmically and using (8) we get

$$\frac{D^{n+p}f}{D^{n+p-1}f} - \frac{1}{2} = \frac{1}{2p}u. \tag{9}$$

Again taking the logarithmic derivative of (9) and using (8) we have

$$\frac{D^{n+p+1}f}{D^{n+p}f} - \frac{1}{2} = \frac{1}{n+p+1}\left(\frac{1}{2} + \frac{n+p}{2p}u + \frac{zu'}{u+p}\right). \tag{10}$$

Since $f \in K_{n+p}$, therefore

$$\operatorname{Re}\left(\frac{n+p}{2p}u + \frac{zu'}{u+p}\right) > -\frac{1}{2}, \qquad |z| < R. \tag{11}$$

In order to prove the theorem it is sufficient to prove that $\operatorname{Re} u > 0$ for $|z| < 1$. Let $u = p(1+w)/(1-w)$; then

$$\frac{zu'}{u+p} + \frac{n+p}{2p}u = \frac{zw'}{w}\cdot\frac{w}{1-w} + \frac{n+p}{2}\cdot\frac{1+w}{1-w}. \tag{12}$$

If $\operatorname{Re} u(z_0) = 0$ for a certain z_0, $|z_0| < R$, and $\operatorname{Re} u(z) \geqslant 0$, $|z| \leqslant |z_0|$, then $|w(z)| \leqslant |w(z_0)| = 1$ for $|z| \leqslant |z_0|$ and $w(z_0) \neq 1$. By applying Lemma 1, (12) gives

$$\operatorname{Re}\left(\frac{z_0 u'(z_0)}{u(z_0) + p} + \frac{n+p}{2p} u(z_0)\right) = -\frac{1}{2} \frac{z_0 w'(z_0)}{w(z_0)} \leqslant -\frac{1}{2},$$

which contradicts (11). Therefore $\operatorname{Re} u(z) > 0$ for $|z| < R$, and hence $g(z)$ is p-valent in $|z| < R$. Thus we see that $g(z)$ cannot vanish on $|z| = R$ if $R < 1$. Hence $R = 1$ and this proves the theorem.

REMARK 1. If we put

$$g(z) = z^p\left(\frac{D^{n+p-1}f}{z^p}\right)^{p/(n+p)}, \tag{13}$$

then we have

$$p\frac{D^{n+p}f}{D^{n+p-1}f} = \frac{zg'}{g}.$$

This implies that $f \in K_{n+p-1}$ if and only if g is p-valently starlike of order $\frac{1}{2}$.

3. Special elements of K_{n+p-1}. In this section, we form special elements of the class K_{n+p-1} via the Hadamard product of elements of K_{n+p-1} and $h_\nu(z)$, where

$$h_\nu(z) = \sum_{j=p}^{\infty} \frac{\nu + p}{\nu + j} z^j, \qquad \operatorname{Re} \nu > -p. \tag{14}$$

THEOREM 2. *If p is any positive integer, n an integer greater than $-p$ and $\operatorname{Re} \nu \geqslant (n-p)/2$, then $f * h_\nu \in K_{n+p-1}$ for all $f \in K_{n+p-1}$. In particular $h_\nu \in K_{n+p-1}$.*

PROOF. One can easily verify that the function $F = f * h_\nu$ satisfies

$$z(D^{n+p-1}F)' = (p+\nu)D^{n+p-1}f - \nu D^{n+p-1}F. \tag{15}$$

Since the families K_{n+p-1} are compact, it is enough to prove the theorem for $\operatorname{Re} \nu > (n-p)/2$. Let

$$g = z^p\left(\frac{D^{n+p-1}F}{z^p}\right)^{p/(n+p)}, \tag{16}$$

and $R = \sup\{r\colon\ g(z) \neq 0,\ 0 < |z| < r < 1\}$. We shall prove that $\operatorname{Re}(zg'/g) > p/2$, $|z| < R$, and this implies $R = 1$. Define a regular function $w(z)$ in E by

$$p\frac{D^{n+p}F}{D^{n+p-1}F} = \frac{zg'}{g} = \frac{p}{1 - w(z)}. \tag{17}$$

Obviously $w(0) = 0$, $w(z) \neq 1$ for $z \in E$. Differentiating (16) logarithmically and using (15) we obtain, after a simple computation,

$$\left(\frac{n+p}{1-w} + \nu - n\right)D^{n+p-1}F = (p+\nu)D^{n+p-1}f. \tag{18}$$

Again, differentiating (18) logarithmically and using (8) and (15), we get

$$\frac{D^{n+p}f}{D^{n+p-1}f} = \frac{zw'}{(1-w)^2}\left(\frac{n+p}{1-w} + \nu - n\right)^{-1} + \frac{1}{1-w}. \tag{19}$$

We show $|w(z)| < 1$, for otherwise by Lemma 1, there exists z_0 in $|z| < R$ such that $z_0 w'(z_0) = kw(z_0)$, $|w(z_0)| = 1$ and $k \geqslant 1$. Then

$$\frac{D^{n+p}f(z_0)}{D^{n+p-1}f(z_0)} = \frac{1}{1 - w(z_0)} + \frac{kw(z_0)}{(1 - w(z_0))^2} \cdot \left(\frac{n+p}{1 - w(z_0)} + \nu - n\right)^{-1}.$$

Since Re $1/(1 - w(z_0)) = \frac{1}{2}$, $w(z_0)/(1 - w(z_0))^2$ is real and negative and

$$\operatorname{Re}\left(\frac{n+p}{1 - w(z_0)} + \nu - n\right)^{-1} > 0 \quad \text{for } \operatorname{Re} \nu > \frac{n-p}{2},$$

hence

$$\operatorname{Re} \frac{D^{n+p}f(z_0)}{D^{n+p-1}f(z_0)} < \frac{1}{2}.$$

This contradicts the fact that $f \in K_{n+p-1}$. Hence $F \in K_{n+p-1}$. Since $h_\nu = (z^p/(1 - z)) * h_\nu$, therefore $h_\nu \in K_{n+p-1}$.

REMARK 2. The first part of Theorem 2 can also be written in a more suggestive form as follows. Let Re $\nu \geqslant (n - p)/2$. If $f \in K_{n+p-1}$, then

$$F(z) = \frac{\nu + p}{z^\nu} \int_0^z t^{\nu-1} f(t)\, dt \in K_{n+p-1} \quad \text{for } \operatorname{Re} \nu > \frac{n-p}{2}.$$

For $n = 0, p = 1$, we get the result due to Libera [**2**].

THEOREM 3. *Let $F \in K_{n+p-1}$, $t = \operatorname{Re} \nu \geqslant (n - p)/2$. Let $f(z)$ be the unique solution of $F(z) = h_\nu * f(z)$. Then* Re $(D^{n+p}f/D^{n+p-1}f) > \frac{1}{2}$ *in $|z| < R$, where R is the smallest positive root of $(t - n)r^2 + (p + n + 2)r - p - t = 0$.*

This result is sharp when ν is real.

PROOF. Since $F \in K_{n+p-1}$, it follows by Remark 1 that

$$g = z^p \left(\frac{D^{n+p-1}F}{z^p}\right)^{p/(n+p)} \tag{20}$$

is p-valently starlike of order $\frac{1}{2}$. Define a regular function $u(z)$ in E by

$$p\frac{D^{n+p}F}{D^{n+p-1}F} = \frac{zg'}{g} = pu(z). \tag{21}$$

Obviously $u(0) = 1$ and Re $u(z) > \frac{1}{2}$ for $z \in E$. As in Theorem 2, we can easily see that

$$(p + \nu)D^{n+p}f(z) = ((\nu - n)u(z) + (n + p)u^2(z) + zu'(z))D^{n+p-1}F(z), \tag{22}$$

and

$$(p + \nu)D^{n+p-1}f(z) = ((\nu - n) + (n + p)u(z))D^{n+p-1}F(z). \tag{23}$$

From (22) and (23), we get

$$\frac{D^{n+p}f(z)}{D^{n+p-1}f(z)} - \frac{1}{2} = u(z) - \frac{1}{2} + \frac{zu'(z)}{\nu - n + (n + p)u(z)}. \tag{24}$$

It is well known that for $|z| = r < 1$,

$$|zu'(z)| \leqslant (2r/(1 - r^2))\left(\operatorname{Re} u(z) - \tfrac{1}{2}\right). \tag{25}$$

Thus from (24) and (25) we have for $|z| = r < 1$,

$$\operatorname{Re}\left(\frac{D^{n+p}f(z)}{D^{n+p-1}f(z)} - \frac{1}{2}\right) \geqslant \operatorname{Re}\left(u(z) - \frac{1}{2}\right)\left(1 - \frac{2r}{(1-r)(p+t+(t-n)r)}\right), \tag{26}$$

where we have used $\operatorname{Re} u(z) > 1/(1+r)$ for $|z| = r$.

Thus $\operatorname{Re}(D^{n+p}f(z)/D^{n+p-1}f(z) - \frac{1}{2}) > 0$ if the right-hand side of (26) is positive, which is satisfied provided $|z| < R$, where R is the smallest positive root of $(t-n)r^2 + (p+n+2)r - p - t = 0$. To show that the result is sharp, we consider $F(z) = z^p/(1-z)$. Then the unique solution of $F(z) = h_\nu(z) * f(z)$ is

$$f(z) = \frac{z^p(p+\nu-(p+\nu-1)z)}{(p+\nu)(1-z)^2}.$$

Now $F \in K_{n+p-1}$ and it can be easily seen that

$$(z^\nu D^{n+p}F(z))' = (p+\nu)z^{\nu-1}D^{n+p}f(z),$$

and

$$(z^\nu D^{n+p-1}F(z))' = (p+\nu)z^{\nu-1}D^{n+p-1}f(z).$$

This implies that

$$\frac{D^{n+p}f(z)}{D^{n+p-1}f(z)} = \left(\frac{z^{\nu+p}}{(1-z)^{n+p+1}}\right)' \Big/ \left(\frac{z^{\nu+p}}{(1-z)^{n+p}}\right)'$$
$$= \frac{1}{1-z} + \frac{z}{(1-z)(\nu+p+(n-\nu)z)}.$$

This yields

$$\frac{D^{n+p}f(z)}{D^{n+p-1}f(z)} = \frac{1}{2} \quad \text{for } z = -R.$$

Hence the result is sharp for ν real.

References

1. I. S. Jack, *Functions starlike and convex of order* α, J. London Math. Soc. (2) **3** (1971), 469–474.
2. R. J. Libera, *Some classes of regular univalent functions*, Proc. Amer. Math. Soc. **16** (1965), 755–758.
3. S. Ruscheweyh, *New criteria for univalent functions*, Proc. Amer. Math. Soc. **49** (1975), 109–115.
4. T. Umezawa, *Multivalently close-to-convex functions*, Proc. Amer. Math. Soc. **8** (1957), 869–874.

Department of Mathematics, Punjabi University, Patiala-147002, India