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NOTE ON A THEOREM OF BERLINSKII

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ABSTRACT. If a quadratic differential system has four singular points, these are elementary and the sum of their indices is 0 iff the quadrilateral with vertices at the singular points is convex; otherwise the sum of indices is 2 or −2. These facts and the relative positions of the two kinds of singular points are readily proved by consideration of the pencil of isoclines of the system. The theorem is originally due to Berlinskii.

We refer to [1]. In that paper Coppel presents a theorem of Berlinskii with a proof by Kukles and Casanova. We give a simpler proof of purely geometrical character.

THEOREM. *Suppose that a real quadratic differential system has four critical points in the affine plane. If the quadrilateral with vertices at these points is convex then two opposite critical points are saddles and the other two are antisaddles (nodes, foci or centers). But if the quadrilateral is not convex, then either the three exterior points are saddles and the interior vertex an antisaddle or the exterior vertices are antisaddles and the interior vertex a saddle.*

Preliminaries. A real quadratic differential system is one of the form

$$\dot{x} = P(x,y), \qquad \dot{y} = Q(x,y),$$

where $P(x,y)$ and $Q(x,y)$ are real second degree polynomials. We assume that the curves $P = 0$ and $Q = 0$ meet at four distinct points of the affine plane, and then, the four points are simple critical points also known as elementary singular points of the system. That is because the condition of four distinct points ensures that the tangents to $P = 0$ and $Q = 0$ at each of them do not coincide, so that the determinant $P_x(x_i, y_i) \times Q_y(x_i, y_i) - P_y(x_i, y_i) \times Q_x(x_i, y_i)$ is different from zero, $i = 1, 2, 3, 4$. But this is the defining condition of an elementary singular point of the system and it is known that such point is either a saddle or an antisaddle: focus, center, node (see [2, Chapter 6, §3]). The index of a saddle (angular variation of the field along a simple closed curve surrounding the singular point and no other) is -1, and antisaddles have index $+1$. In the course of the proof we will need the fact that the sum of indices of a set of singular points is equal to the angular variation of the field along any simple closed curve surrounding those points and no others, also called the index of the curve with respect to the field (see [2, Appendix, §1]). These four critical points of the system are also base points of a pencil of conics consisting of $P = 0$ and the curves $\lambda P - Q = 0$, with λ a real

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parameter. Every point of the plane, not a base point of the pencil, lies in one and only one conic of the pencil and along every such curve, the field has constant direction λ. (It is parallel to the y-axis on $P = 0$.) This family of conics is the family of *isoclines* of the system. We shall use the degenerate conics of the pencil (those composed of straight lines): their existence is obvious: if $P = 0$ or $Q = 0$ is not already degenerate, choosing a point of the plane collinear with two base points and substituting its coordinates in the equation of the pencil, we get the value of λ for the degenerate conic joining those points. Finally, the degenerate conics of the pencil, if not composed of parallel lines, are self-intersecting, that is, they present double points. We shall establish that two singular points of the system, consecutive on an isocline, have opposite indices unless there is a double point of that isocline between them.

PROOF. Let us establish the following fact: if A and B are singular points of our system, consecutively situated on an isocline, and there are no double points of that isocline between A and B, then $\operatorname{Ind} A = -\operatorname{Ind} B$. It is clear that we can surround A and B by a simple closed curve Γ not surrounding other critical points. Also, referring to Figure 1, we see that the field vector of the isocline, *is* between A and B *the negative* of the field vector on the complementary part of that isocline, near A or B. This is so because isoclines are not tangent to each other so that the vector (P, Q) changes to $(-P, -Q)$ when passing from one side of a singular point to the other along *any* isocline (near A, for instance, we have points of the isocline belonging to $P > 0$ on one side, and to $P < 0$ on the other, and the same applies to Q). This fact, together with our assumption that there are no double points of the isocline surrounded by Γ, allows us to say that the curve Γ has no points in common with the set of points of the plane where the field vector is V_0, as in Figure 1. This implies that the angular variation around Γ ($\operatorname{Ind} \Gamma$) must be zero and $\operatorname{Ind} A = -\operatorname{Ind} B$.

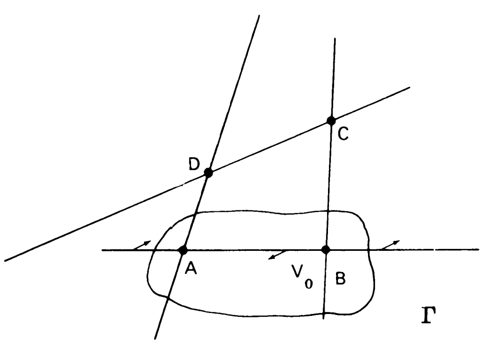

FIGURE 1

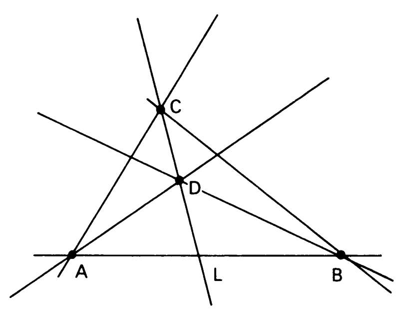

FIGURE 2

The convex case. We consider the degenerate conics of the pencil of isoclines, (AB, CD) and (AD, BC), with no double points inside the quadrilateral. Then (as in Figure 1) A and B are consecutive on that isocline and $\operatorname{Ind} A = -\operatorname{Ind} B$. Similarly $\operatorname{Ind} C = -\operatorname{Ind} B = \operatorname{Ind} A$, and $\operatorname{Ind} D = -\operatorname{Ind} A$.

The nonconvex case. We suppose that the interior vertex of the quadrilateral is D. Take now the isocline (AB, CD). It is clear that D and C are consecutive on this isocline and there are no double points between C and D. This implies Ind $D =$ $-$ Ind C (one can see Figure 2). Similarly A is consecutive with D on (AD, BC) and Ind $D = -$ Ind A. Also on (AC, BD) one sees that Ind $D = -$ Ind B. Summarizing: the three exterior vertices have index of opposite sign to that of the interior vertex. (Of course A and B are consecutive but there is L, a double point, between them.)

References

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