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EXISTENCE THEOREMS FOR GENERALIZED HAMMERSTEIN EQUATIONS

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ABSTRACT. In this paper we obtain existence theorems for generalized Hammerstein-type equations $K(u)Nu + u = 0$, where for each u in the dual X^* of a real reflexive Banach space X, $K(u)$: $X \to X^*$ is a bounded linear map and N: $X^* \to X$ is any map (possibly nonlinear). The method we adopt is totally different from the methods adopted so far in solving these equations. Our results in the reflexive space generalize corresponding results of Petry and Schillings.

Introduction. Let X be a real reflexive Banach space with dual X^*. For each $u \in X^*$, let $K(u)$: $X \to X^*$ be a linear operator and let N be any map (possible nonlinear) from X^* into X. In this paper we establish existence theorems for the "generalized Hammerstein equation"

$$K(u)N(u) + u = 0. \tag{1}$$

Equation (1) may be considered as a generalization of the "Hammerstein equation"

$$KN(u) + u = 0 \tag{2}$$

where the operator K is independent of the solution u. Existence theorems for (2) have been established by many authors; see [2] and [4] for details.

Generalized Hammerstein equations have recently been studied by Avramescu [1], Petry [7], [10], Schillings [12], [13] and Srikanth [14].

In the present note, equation (1) will be considered under monotonicity conditions on N and the family $\{K(u)\}$. Also we impose suitable compactness assumptions on the family $\{K(u)\}$. The theorems we establish below generalize the results of Petry [7] and some results of Schillings [13]. For applications see [7], [12], [13] and [6] and [14].

Throughout this paper we use "$\to$" to denote strong convergence and "$\rightharpoonup$" to denote weak convergence. Also we use $B[0, r]$ to denote the closed ball of radius "r" around the origin and $\dot{B}[0, r]$ to denote the boundary.

Some definitions and results from monotone theory. T: $X \to X^*$ is said to be monotone, if for each $u,v \in X$, $\langle Tu - Tv, u - v\rangle \geqslant 0$, where $\langle\ ,\ \rangle$ denotes the usual bilinear form. T is said to be strictly monotone, if for each $u,v \in X$, $\langle Tu - Tv, u - v\rangle \geqslant 0$ for $u \neq v$. T is said to be coercive if $\langle Tx, x\rangle/\|x\| \to \infty$ as $\|x\| \to \infty$. It is said to be bounded if it maps bounded sets into bounded sets. It is said to be compact if it maps bounded sets into precompact sets.

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DEFINITION 1. $T: X \to X^*$ is said to be of type (M) if the following hold:

(i) If $x_n \rightharpoonup x$ in X, $T(x_n) \rightharpoonup g$ in X^* and

$$\lim_n \sup \langle Tu_n, x_n \rangle = \langle g, x \rangle,$$

then $T(x) = g$.

(ii) T is continuous from finite-dimensional subspaces of X into X^* equipped with the weak topology (i.e. T is finitely continuous).

DEFINITION 2. $T: X \to X^*$ is said to be of type $(S+)$ if the following holds:

If $u_n \rightharpoonup u$ in X, Tu_n approaches the corresponding sequence in X^* and

$$\lim_n \sup \langle Tu_n, u_n - u \rangle \leqslant 0,$$

then $u_n \to u$ and $Tu_n \rightharpoonup Tu$.

The relation between finitely continuous maps of type $(S+)$ and maps of type (M) is evident.

PROPOSITION 1 ([5], [11]). *Let X be a real reflexive Banach space and T be a bounded mapping of type (M) from X into X^*. Suppose that the mapping T is coercive, then $R(T) = X^*$.*

REMARK 1. In the above, one can replace the coercivity by a weaker condition $\langle Tu, u \rangle > 0$ for all $u \in B[0, r]$ for some $r > 0$ and obtain an existence result for $Tu = 0$.

Statement of the main theorem. Before we state the main theorem we make the following assumptions:

ASSUMPTION A.

(1) X is a real reflexive Banach space with dual X^*.

(2) (a) For each u in X^*, $K(u): X \to X^*$ is a bounded linear map.

(b) $\langle K(u)v, v \rangle \geqslant 0$ for all u in X^* and v in X.

(c) $\sup_{u \in X^*} \|K(u)\| \leqslant \alpha$ for some $\alpha > 0$.

(d) $K(\cdot)(\cdot)$ is continuous from $X^* \times X$ into X^*.

(e) $K(\cdot)v$ is compact for each fixed v.

THEOREM 1. *Let assumption* (A) *hold and let $N: X^* \to X$ be a bounded, continuous, monotone map of type $(S+)$. Suppose there exists a constant β such that*

$$\|Nu\| \leqslant \beta \|u\| \quad \text{for } \|u\| \geqslant r, \tag{3}$$

where $r > 0$ is some real number and $\alpha\beta < 1$. Then $K(u)N(u) + u = 0$ has a solution.

Before we prove the theorem we prove the following lemma and proposition required in the proof of the theorem.

LEMMA 1. *Let X be a real reflexive Banach space with dual X^*. Suppose that N: $X \to X^*$ is a one-to-one, continuous map of type $(S+)$. Then, for every sequence u_n in X^* such that*

(i) $u_n \rightharpoonup u$ in X^*;

(ii) $N^{-1}(u_n)$ is bounded in X;

(iii) $\lim_n \sup\langle N^{-1}u_n, u_n - u\rangle \leq 0$,

we have $u_n \to u$ in X^*.

PROOF. Let $u_n \rightharpoonup u_0$ in X^* and $N^{-1}u_n$ be bounded with

$$\lim_n \sup\left\langle N^{-1}u_n, u_n - u_0\right\rangle \leq 0.$$

Since $N^{-1}u_n$ is bounded and the space is reflexive $N^{-1}u_n \rightharpoonup w_0$ in X (actually we have a subsequence converging weakly to w_0, which we shall not distinguish).

Hence we have $\lim_n \sup\langle N^{-1}u_n - w_0, u_n - u_0\rangle \leq 0$. Since N is of type $(S+)$ this implies $N^{-1}u_n \to w_0$ and by the continuity of N, we have $u_n \to Nw_0$. Since $u_n \rightharpoonup u_0$ this implies $Nw_0 = u_0$ and $u_n \to u_0$.

ASSUMPTION B.

(1) X is a real reflexive Banach space with dual X^*.

(2) (a) For each u in X^*, $K(u)$: $X \to X^*$ is any map (not necessarily linear).

(b) For each u in X^*, $K(u)$ is monotone.

(c) For each bounded set $A \subset X$ and $B \subset X^*$, there exists a constant A_B such that $\|K(u)v\| \leq A_B$ for all u in B and v in A.

(d) $u_n \to u$ in X^* and $v_n \to v$ in X implies $K(u_n)v_n \rightharpoonup K(u)v$.

(e) For each fixed v in X the map $K(\cdot)v$: $X^* \to X^*$ is compact.

PROPOSITION 2. *Let Assumption* (B) *hold and let N: $X^* \to X$ be a one-to-one, onto and continuous map of type $(S+)$ with N^{-1} bounded. Suppose further that there exist two real valued functions C_1 and C_2 defined on R^+ such that $C_1(r) + C_2(r) \to \infty$ as $r \to \infty$ and for each u in X,*

$$\text{(i)} \quad \langle K(u)v, v\rangle \geq C_1(\|v\|)\|v\| \quad \forall v \text{ in } X, \tag{4}$$

and

$$\text{(ii)} \quad \langle N^{-1}v, v\rangle \geq C_2(\|v\|)\|v\| \quad \forall v \text{ in } X.$$

Then the equation $K(u)N(u) + u = 0$ has a solution.

PROOF. We first note that under the above hypothesis, solving the equation $K(u)N(u) + u = 0$ is equivalent to solving the equation $K(N^{-1}v)v + N^{-1}v = 0$ for some v in X.

Define a map S: $X \to X^*$ as follows. For v in X, let

$$S(v) = K(N^{-1}v)v + N^{-1}v.$$

From our hypothesis it is obvious that S is well defined. Further we claim that the map S is of type $(S+)$.

Suppose that $v_n \rightharpoonup v_0$ in X and $\lim_n \sup\langle Sv_n, v_n - v_0\rangle \leq 0$, i.e.

$$\lim_n \sup\left\langle K\left(N^{-1}v_n\right)v_n + N^{-1}v_n, v_n - v_0\right\rangle \leq 0.$$

By assumptions B(b) and B(e) it follows that for some subsequence which we shall not distinguish

$$\limsup_n \langle N^{-1}v_n, v_n - v_0 \rangle \leqslant 0.$$

By Lemma 1, it now follows that $v_n \to v_0$. In fact, we can show that every subsequence of the original sequence v_n has a strongly convergent subsequence converging to v_0. Hence $v_n \to v_0$. Moreover a look at the proof of Lemma 1 ensures that $N^{-1}v_n \to N^{-1}v_0$. Hence by assumption B(d) it follows that

$$K(N^{-1}v_n)v_n + N^{-1}v_n \rightharpoonup K(N^{-1}v_0)v_0 + N^{-1}v_0.$$

Also it is obvious that S is bounded and finitely continuous. Hence S is a bounded finitely continuous map of type $(S+)$. Now the required result follows from our assumption (4) and Proposition 1. In fact, under the assumptions of the above proposition S is onto.

PROOF OF THE THEOREM. By a recent result of Troyanski **[15]**, each reflexive Banach space X has an equivalent norm in which both X and X^* are locally uniformly convex, hence we may assume without loss of generality that X and X^* are locally uniformly convex. Hence if J denotes the (single-valued) normalized duality mapping of X^* into X given by $Ju = w$, where $\langle Ju, u\rangle = \|Ju\|\,\|u\|$ and $\|Ju\| = \|u\|$ then J is a continuous map from the strong topology of X^* to the strong topology of X. Also J is strictly monotone and of type $(S+)$ (see **[3]**). Hence for each $\lambda > 0$, $(N + \lambda J)$ is of type $(S+)$ whenever N is monotone. We use these facts in the proof.

Define N_n: $X^* \to X$, for each positive integer n by $N_n u = Nu + Ju/n$. One can easily verify that N_n satisfies the assumption on N of Proposition 2. Further by the assumption (3) of the theorem, N_n^{-1} is coercive (this is easy to prove and we omit the details).

The family $\{K(u)\}$ and N_n satisfy all the requirements of Proposition 2. Hence the equation $K(u)N_n u + u = 0$ has a solution for each positive integer n. Let one such solution be u_n, i.e. $K(u_n)N_n u_n + u_n = 0$.

We claim that the sequence $\{u_n\}$ as n varies over the set of positive integers is bounded. Suppose not, then

$$\|u_n\| = \|K(u_n)N_n u_n\| \leqslant \alpha\|N_n u_n\| \leqslant \alpha(\beta + 1/n)\|u_n\|. \tag{5}$$

Since we can choose n large, such that $\alpha(\beta + 1/n) < 1$, we have a contradiction from equation (5) above.

Since the space is reflexive and $\{u_n\}$ is a bounded sequence it has a weakly convergent subsequence $u_n \rightharpoonup u_0$ from some u_0 in X^*. We claim $K(u_0)Nu_0 + u_0 = 0$. To show this, we first show that $u_n \to u_0$ (a subsequence of u_n will be shown to converge to u_0 strongly, but we shall not distinguish this subsequence).

Since $u_n \rightharpoonup u_0$ and N is bounded we can assume $Nu_n \rightharpoonup v_0$ for some $v_0 \in X$.

Consider

$$\begin{aligned}\langle Nu_n, u_n\rangle &= \langle v_0, u_n\rangle + \langle Nu_n - v_0, u_n\rangle\\ &= \langle v_0, u_n\rangle + \langle Nu_n - v_0, -K(u_n)N_n u_n\rangle\\ &= \langle v_0, u_n\rangle + \langle Nu_n - v_0, -K(u_n)Nu_n\rangle - (1/n)\langle Nu_n - v_0, K(u_n)Ju_n\rangle\\ &= \langle v_0, u_n\rangle + \langle Nu_n - v_0, -K(u_n)Nu_n\rangle - (1/n)\langle Nu_n - v_0, K(u_n)Ju_n\rangle\\ &= \langle v_0, u_n\rangle + \langle Nu_n - v_0, -K(u_n)Nu_n + K(u_n)v_0\rangle\\ &\quad + \langle Nu_n - v_0, -K(u_n)v_0\rangle - (1/n)\langle Nu_n - v_0, K(u_n)Ju_n\rangle\end{aligned}$$

Since each $K(u)$ is monotone, the above expression leads to,

$$\begin{aligned}\langle Nu_n, u_n\rangle \leqslant \langle v_0, u_n\rangle &+ \langle Nu_n - v_0, -K(u_n)v_0\rangle\\ &- (1/n)\langle Nu_n - v_0, K(u_n)Ju_n\rangle.\end{aligned}$$

The second expression in the right hand side of the above can be shown to tend to zero, as n tends to infinity, by assumption A(e) (actually we have to consider a subsequence which goes to zero). Also the last term on the right hand side goes to zero as $n \to \infty$ because $\langle Nu_n - v_0, K(u_n)Ju_n\rangle$ is bounded. Hence we have

$$\lim_n \sup\langle Nu_n, u_n\rangle \leqslant \langle v_0, u_0\rangle.$$

Since N is of type $(S+)$ this implies $u_n \to u_0$. Now $K(u_0)Nu_0 + u_0 = 0$ follows because it is the weak limit of $K(u_n)N_n u_n + u_n = 0$. Hence the result.

REMARK 2. If in the Theorem 1 instead of assumption (3) one assumes the condition that $\langle Nu, u\rangle \geqslant 0$ outside some ball of radius r, then we can obtain the existence result for $K(u)N(u) + u = 0$ without assumption A(c).

PROOF OF REMARK 2. Consider $N_n u = Nu + Ju/n$. Then $\langle N_n^{-1}v, v\rangle \geqslant 0$ for $\|v\| \geqslant R$ where R is some real number greater than $R_0 = \sup_{\|u\|<r} \|Nu\|$. Now in Proposition 2 use Remark 1 to obtain a solution for $K(u)N_n u + u = 0$. The rest of the remark follows as in the Theorem.

REMARK 3. Schillings obtained the result as in Remark 2 in **[13]**, but he has demicontinuity on N and also he assumed that the map $u \to K(u)$ from X^* into $L(X^*, X)$ is compact and continuous. If however in Theorem 1 we assume N to be just demicontinuous then we would require only $u \to K(u)$ to be continuous from X^* into $L(X^*, X)$ in addition to our assumption, some of which follow the above mentioned continuity. Further, it is important to note that Schillings wanted N to be strongly monotone and the space to be separable reflexive in order to obtain an existence result for $K(u)Nu + u = 0$ if the conditions on the family $K(u)$ are as in our theorem (see [**13**, Theorems 1 and 3]).

REMARK 4. The assumption of linearity on each $K(u)$ can also be omitted by slightly changing the proof of Theorem 1 (see [**14**]).

REMARK 5. For some more results on generalized Hammerstein equations see [**14**], where results much stronger than the one mentioned in the remark [**13**, p. 186] are obtained.

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