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A DIVERGENT, TWO-PARAMETER, BOUNDED MARTINGALE

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ABSTRACT. An example is given of a divergent, uniformly bounded martingale $X = \{X_t\colon t \in T\}$ where the index t ranges over the set T of pairs of positive integers with the usual coordinatewise ordering.

This note offers an example of a divergent, uniformly bounded, two-parameter martingale which supplements the infinite-parameter example of Dieudonné [3]. Also offered here is a divergent, uniformly bounded, two-parameter, reversed martingale simpler than the one mentioned in [4].

For m a positive integer and $0 < \varepsilon < 1$ an (m, ε)-*daisy* is a partition Π of the universal event consisting of $m + 1$ events $C, B_1, \ldots, B_m$ where C, the *center* of the daisy, has probability ε, and the B_i have equal probability $(1 - \varepsilon)/m$. Let Π_i be the two-element partition consisting of $C \cup B_i$ and its complement. Plainly, the value of $P(C|\Pi_i)$ on $C \cup B_i$ is $(1 + (1 - \varepsilon)/m\varepsilon)^{-1}$, which is now abbreviated to $c(\varepsilon, m)$. Consequently, $\sup_{1 < i < m} P(C|\Pi_i) = c(\varepsilon, m)$ everywhere. Indeed, a simple calculation shows that, for any pair s of positive integers a, b,

$$\sup_{a < i < m-b} P(C|\Pi_i) = c(\varepsilon, m) \tag{1}$$

with probability greater than $1 - |s|/m$, where $|s|$ is $a + b$.

Let T be the set of all ordered couples of positive integers endowed with the coordinatewise ordering, that is, $s \leqslant t$ if each coordinate of $t - s$ is nonnegative.

An array of partitions Π_t, $t \in T$, is *based on* the (m, ε)-daisy Π if $|t| = m$ and $t = (i, j)$ implies that Π_t is Π_i, and if Π is a refinement of each Π_t.

Let $\Pi^1, \Pi^2, \ldots$ form a mutually independent sequence of partitions of the universal event of a suitable probability space, such that, for each r, Π^r is an (m_r, ε_r)-daisy. Let $\{(\Pi^r)_t, t \in T\}$ be an array of partitions based on Π^r, and let $\mathcal{S}_t$ be the sigma-field generated by the partitions $(\Pi^r)_t$, $r = 1, 2, \ldots$. Let A be the union of the centers C^r of the daisies Π^r.

LEMMA. *If $m_r \varepsilon_r \to \infty$, then for each s*

$$\sup_{t > s} P(A|\mathcal{S}_t) = 1 \quad \textit{almost surely.} \tag{2}$$

PROOF. As is evident from (1), $m_r \varepsilon_r \to \infty$ implies that, for each s, $\sup_{t>s} P(C^r|(\Pi^r)_t) \to 1$ in distribution as $r \to \infty$. Thus

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$$\sup_r \sup_{t>s} P(C^r|(\Pi^r)_t) = 1 \quad \text{almost surely.}$$

Since A includes C^r, $P(A|\mathbb{S}_t)$ exceeds $P(C^r|\mathbb{S}_t)$, which in turn equals $P(C^r|(\Pi^r)_t)$ because, for each t, the partitions $(\Pi^1)_t, (\Pi^2)_t, \ldots$ are independent. Consequently, (2) must hold.

An array $\{\Pi_t, t \in T\}$ is *decreasing* if $s \leqslant t$ implies Π_s is a refinement of Π_t. To obtain a decreasing array based on an (m, ε)-daisy Π, first note that Π_t is determined for $|t| = m$. Then Π_t must be the trivial partition for $|t| > m$, and the array can be completed in various ways, for example by setting $\Pi_t = \Pi$ for $|t| < m$. Say the *decreasing case* obtains if for each r the array $\{(\Pi^r)_t, t \in T\}$ introduced above is decreasing. The *increasing case* is defined analogously.

Plainly, $P(A|\mathbb{S}_t)$ is a uniformly bounded martingale or reversed martingale according as the increasing or decreasing case obtains.

PROPOSITION. *Suppose $m_r\varepsilon_r \to \infty$ and $\Sigma\varepsilon_r < \infty$. Then, in the increasing case, $P(A|\mathbb{S}_t)$, $t \in T$, diverges with positive probability and, in the decreasing case, it diverges with probability one.*

PROOF. Since the centers C^r are independent, $\Sigma\varepsilon_r < \infty$ implies $0 < PA < 1$. Consider first the decreasing case. For any increasing sequence $t(j) \in T$, $\bigcap \mathbb{S}_{t(j)}$ is part of the trivial tail sigma-field of the independent sequence of partitions $\Pi^1, \Pi^2, \ldots$ because $m_r \to \infty$ and $(\Pi^r)_t$ is the trivial partition for $|t| > m_r$. Thus $P(A|\mathbb{S}_{t(j)})$ converges almost surely to the constant $PA < 1$. This, together with (2), implies that $P(A|\mathbb{S}_t)$ diverges almost surely. Consider now the increasing case. If $|t| > m_r$, then C_r, the center of Π^r, is $(\Pi^r)_t$-measurable, and hence $\mathbb{S}_t$-measurable. So if $t(j) = (j, j)$, each C^r and, hence, A, is measurable relative to the limit of the $\mathbb{S}_{t(j)}$. Therefore, by Levy's martingale convergence theorem, $P(A|\mathbb{S}_{t(j)})$ converges to zero almost everywhere off A (and to 1 almost everywhere on A). This, together with (2), implies that $P(A|\mathbb{S}_t)$ diverges almost everywhere on the complement of A. □

As is easily verified, a uniformly bounded martingale parameterised by T which diverges almost surely is $M_t = \Sigma M_t^n/2^n$, where $(M_t^1), (M_t^2), \ldots$ is a sequence of independent copies of the martingale described above.

Of course, examples such as these indicate the necessity of some auxiliary condition to guarantee the almost sure convergence of multi-parameter martingales. The last word on this subject does not yet seem to have been said, but some such supplementary conditions can be found in the references.

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