FINITENESS OF ISOTOPY CLASSES OF CERTAIN KNOTS

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ABSTRACT. We prove finiteness results for the isotopy classes of certain high-dimensional knots whose complement satisfies some homology conditions. In some particular cases, the isotopy class of a knot is determined by the twisted homology of its complement.

1. Introduction. An n -dimensional knot is an oriented submanifold N^n of the sphere S^{n+2} which is homeomorphic to S^n . (We work in the PL-category and submanifolds are locally flat.) Two knots N_0 and N_1 are called isotopic if there is a homeomorphism $H: S^{n+2} \times I \rightarrow S^{n+2} \times I$ which is orientation and level preserving so that $H(x, 0) = (x, 0)$ and $H|_{N_0 \times \{1\}}$ is an orientation preserving homeomorphism from $N_0 \times \{1\}$ onto $N_1 \times \{1\}$. The complement $X(N)$ of a knot N is defined by $X(N) = S^{n+2} - N$ and its universal covering is denoted by $\tilde{X}(N)$.

In this paper, we are interested in n -dimensional knots N such that

(a) $\pi_1(X(N)) \simeq \mathbb{Z}$.

(b) $H_i(\tilde{X}(N)) = 0$ for $i \neq k$ and $i \leq (n+1)/2$.

Conditions (a) and (b) determine completely $H_*(\tilde{X}(N))$ by the Poincaré-Blanchfield duality [Le, Theorem 3.4].

Let Λ be the group ring $\mathbb{Z}[\mathbb{Z}]$ which is isomorphic to the ring $\mathbb{Z}[t, t^{-1}]$ of Laurent polynomials in one variable t . The group $A = H_k(\tilde{X}(N))$ for a knot satisfying (a) and (b) must be a Λ -module of type K , i.e., a Λ -module satisfying

(1) A is finitely generated over Λ .

(2) The multiplication by $(1-t)$ is an isomorphism.

(See [Ke, II, Theorem II.1] for a proof.) On the other hand, for any Λ -module A of type k and any pair (k, n) of integers such that $2(k+2) \leq n$ ($2(k+1) \leq n$ if A is $\mathbb{Z}$ -torsion-free) there exists an n -dimensional knot N satisfying (a) and (b) with $H_k(\tilde{X}(N)) \simeq A$ (isomorphism of Λ -module). This knot may be obtained by the Kervaire construction ([Ke, Theorem II.1]; see also [Ha, (2.5), p. 193 and Lemme 7.5]).

DEFINITION. An n -dimensional knot satisfying conditions (a) and (b) with $H_k(\tilde{X}(N)) \simeq A$ is called an (A, k, n) -knot.

Theorem 7.4 of [Ha] asserts that for a given triple (A, k, n) satisfying certain conditions, the isotopy classes of (A, k, n) -knots are finite in number. However, except for the special cases indicated in [Ha, p. 194], the required conditions on the triple (A, k, n) are difficult to verify. The aim of this paper is to enrich the results

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of [Ha, §7] by proving that the number of isotopy classes of (A, k, n) -knots is finite under very simple hypotheses.

2. Statement of the main results. Let Γ and Γ_0 be the subset of $\mathbb{N} \times \mathbb{N}$ defined as follows:

$$\Gamma = \{(k, n) \in \mathbb{N} \times \mathbb{N} | 2k + 4 \leq n \leq 3k - 4\},$$

$$\Gamma_0 = \{(k, n) \in \mathbb{N} \times \mathbb{N} | 2k + 2 \leq n \leq 3k - 4\}.$$

So Γ and Γ_0 may be visualized as the subset of points in $\mathbb{R}^2$ with integral coordinate belonging to the following closed hatched areas:

Let A be a Λ -module of type K . By what was recalled in the Introduction, the set of (A, k, n) -knots is not empty when $(k, n) \in \Gamma$ (or $(k, n) \in \Gamma_0$ and A is $\mathbb{Z}$ -torsion-free).

THEOREM 1. *Let A be a Λ -module of type K . Then, for every pair $(k, n) \in \Gamma$, the set of isotopy classes of (A, k, n) -knots is finite.*

THEOREM 2. *Let A be a Λ -module of type K which is $\mathbb{Z}$ -torsion-free. Then for $(k, n) \in \Gamma_0$, the set of isotopy classes of (A, k, n) -knots is finite.*

For an integer m , let $\mathbb{Z}[1/m]$ denote the subgroup of $\mathbb{Q}$ generated by m^{-i} ($i \in \mathbb{N}$).

THEOREM 3. *Let A be a $\mathbb{Z}$ -torsion-free Λ -module of type K and k be an integer. Suppose that $A \otimes_{\mathbb{Z}} \mathbb{Q} \simeq \mathbb{Q}^r$ and that A contains $(\mathbb{Z}[1/k!])^r$ as an abelian subgroup. Then, for any $(k, n) \in \Gamma_0$, there is exactly one isotopy class of (A, k, n) -knots.*

REMARK. As examples of Λ -modules of type K satisfying the condition of Theorem 3 one can take

$$A = \Lambda / ((k!)t^s - (k! + 1))$$

or

$$A = \bigoplus_{s \text{ times}} \Lambda / ((k!)t - (k! + 1)).$$

In fact, both of these Λ -modules are $\mathbf{Z}$ -isomorphic to $(\mathbf{Z}[1/k!(k! + 1)])^s$ by [Sr, Theorems 2.3 and 2.5].

Proofs of Theorems 1 to 3 are given in the next section. They use results on the homotopy groups of certain Moore spaces which are given and proven in §4.

3. Proofs of Theorems 1 to 3. Recall that an n -dimensional knot is antisimple (see [Ha]) if $X(N)$ has a handle decomposition without handles of index

$$\begin{aligned} & -j + 1 && \text{if } n = 2j, \\ & -j + 1 \quad \text{and} \quad j + 2 && \text{if } n = 2j + 1. \end{aligned}$$

Let N be an (A, k, n) -knot with $(k, n) \in \Gamma$. Proposition 6.1 of [Ha] together with the Poincaré-Blanchfield duality [Le, Theorem 3.4] then implies that N is antisimple. Let K be the union of the handles of index $< j - 1$ of a handle decomposition of $X(N)$ satisfying the above condition. This complex K is a homology circle with $\pi_1(K) \simeq \mathbf{Z}$. The “meridian” element of N distinguishes a generator $\alpha \in \pi_1(K)$. The equivalence class of the pair (K, α) is an invariant of the isotopy class of N (see [Ha, Lemme 3.1]; two pairs (K_i, α_i) ($i = 0, 1$) are equivalent if there is a homotopy equivalence $f: K_0 \rightarrow K_1$ such that $\pi_1(f)(\alpha_0) = \alpha_1$).

The complex K collapses to a complex of dimension $j - 1$, which we call K again. Let E_K^{n+1} be a regular neighborhood of K in $\mathbf{R}^{n+1}$. When $\pi_1(K) \simeq \mathbf{Z}$ and $n > 4$, the set of isotopy classes of antisimple knots having (K, α) as invariant pair is in bijection with $[E_K^{n+1}, E_K^{n+1}]_{\text{rel } \partial}$, where the latter denotes the set of homotopy classes of maps $f: E_K^{n+1} \rightarrow E_K^{n+1}$ so that $f|_{\partial E_K^{n+1}} = \text{id}$ (the homotopies are fixed on ∂E_K^{n+1}). This result was proven in [Ha, Theorem 3.3].

Therefore, to prove Theorem 1, it is enough to establish that:

- (1) The set of equivalence classes of pairs (K, α) corresponding to (A, k, n) -knots (for a fixed triple (A, k, n)) is finite.
- (2) For each (K, α) corresponding to an (A, k, n) -knot, the set $[E_K^{n+1}, E_K^{n+1}]_{\text{rel } \partial}$ is finite.

Part (1) was proven in [Ha, Lemme 7.5] (see Remark 4.4). Part (2) will now be proven by obstruction theory. By [Ha, Lemme 7.5], E_K^{n+1} has a handle decomposition with handles of index 0, 1, k , $k + 1$, $k + 2$. The nonzero obstruction groups for isotopying rel ∂ two maps from E_K^{n+1} to itself are $H^i(E_K^{n+1}, \partial E_K^{n+1}; \pi_i(K))$ for $i = n - k - 1, n - k, n - k + 1, n$ and $n + 1$. Hence it is enough to prove that $\pi_i(K)$ is finite in these dimensions. Since we are dealing with (A, k, n) -knots, the universal covering of K is a Moore space $M(A; k)$ (i.e. has only one nonzero homology group $H_k(M(A; k)) \simeq A$). As A is a Λ -module of type K , one has that $A \otimes_{\mathbf{Z}} \mathbf{Q}$ has finite $\mathbf{Q}$ -dimension [Sr, Theorems 2.1 and 2.2] and the $\mathbf{Z}$ -torsion of A is a finite group [Ke, Lemme 11.8]. Hence, by Proposition (4.8), $\pi_i(M(A; k))$ is finite for $k + 1 < i < 3k - 3$ and $i \neq 2k - 1$. Thus we have to prove that none of the integers $n - k - 1, n - k, n - k + 1, n$ or $n + 1$ is equal to $2k - 1$ and that they

are all $\leq 3k - 3$. The reader will check easily that these conditions are guaranteed by the fact that $(k, n) \in \Gamma$. Thus Theorem 1 is proved.

The proof of Theorem 2 is exactly the same as that of Theorem 1 except that the condition $(k, n) \in \Gamma_0$ implies that an (A, k, n) -knot is antisimple when A is $\mathbf{Z}$ -torsion-free. Indeed, if $n \geq 2k + 2$, one gets $H_j(\tilde{X}(N)) = H_{j+1}(\tilde{X}(N)) = 0$ if $n = 2j$ and $H_j(\tilde{X}(N)) = H_{j+1}(\tilde{X}(N)) = H_{j+2}(\tilde{X}(N)) = 0$ if $n = 2j + 1$, using [Le, Theorem 3.4]. Thus N is antisimple by [Ha, Proposition 6.1].

In the same way, Theorem 3 will be proven by establishing that

(1') There is exactly one equivalence class of pairs (K, α) corresponding to (A, k, n) -knots when (A, k) is given and A is $\mathbf{Z}$ -torsion-free.

(2') $[E_K^{n+1}, E_K^{n+1}]_{\text{rel } \partial}$ has one element for K as in (1) and A satisfies the hypothesis of Theorem 3.

Let us prove (1). Let (K, α) be a pair associated with an (A, k, n) -knot with A $\mathbf{Z}$ -torsion-free. By [Ha, Lemme 7.5 and its proof], the homotopy type of K is uniquely determined by (A, k) and contains a complex of the form $S^1 \vee B$ with B simply connected. Such a complex admits a homotopy equivalence switching the generators of the fundamental group. Therefore there is only one equivalence class of these pairs (K, α) .

Finally (2') is proven similarly to (2), by checking that $\pi_i(M(A; k)) = 0$ for $i = n - k - 1, n - k, n - k + 1, n, n + 1$. We just use here Proposition (4.9) instead of Proposition (4.8).

4. Homotopy groups of $M(A; k)$. In this section we prove the results about the homotopy groups of Moore spaces we need in the proof of Theorems 1, 2 and 3 (Propositions (4.8) and (4.9)).

(4.1) If A is an abelian group, its rank $r(A)$ is defined as $r(A) = \dim_{\mathbf{Q}}(A \otimes \mathbf{Q})$. So $r(A) \in \mathbf{N} \cup \{\infty\}$. The $\mathbf{Z}$ -torsion subgroup of A is denoted by $t(A)$.

If P is a set of prime integers, we denote by $\mathbf{Z}_{(P)}$ the subring of $\mathbf{Q}$ of the fractions m/n with $(n, p) = 1$ for $p \in P$. If $m \in \mathbf{N}$, the subgroup $\mathbf{Z}[1/m]$ generated by $1/m^r$, $r \geq 0$, coincides with $\mathbf{Z}_{(P)}$ for $P = \{p \text{ prime} | (p, m) = 1\}$.

The set of all prime integers is ordered with its natural order and p_i denotes the i th prime for this order.

(4.2) We recall here the classification of the subgroups of $\mathbf{Q}$ [Fu, §85]. The isomorphism classes of subgroups of $\mathbf{Q}$ are in bijection with the equivalence classes of sequences (u_n) with $n \in \mathbf{N}$ and $u_n \in \mathbf{N} \cup \{\infty\}$, where (u_n) is equivalent to (u'_n) if $\sum_{i=1}^{\infty} |u_i - u'_i| < \infty$ (with the convention $\infty - \infty = 0$). The sequence (u_n) corresponds to the subgroup of $\mathbf{Q}$ generated by $p_i^{-u_i}$ for $i \leq u_i$. For example, $\mathbf{Z}_{(P)}$ has characteristic sequence

$$u_i = \begin{cases} 0, & \text{if } p \in P, \\ \infty & \text{if } p \notin P. \end{cases}$$

From this classification, we deduce easily the following lemma:

(4.3) LEMMA. Let A be a subgroup of $\mathbf{Q}$ and $(u_n(A))$ its characteristic sequence. Let $P = \{p_i | u_i(A) \neq \infty\}$. Then A is isomorphic to a direct limit of groups isomorphic to $\mathbf{Z}_{(P)}$.

(4.4) REMARK. This lemma is actually necessary to complete the proof of [Ha, Assertion, p. 107]. Also this assertion needs the hypothesis that $r(B)$ is finite. This does not affect the consequences we drew from it (i.e. [Ha, Lemme 7.5]), and which are used in this paper. We leave it to the reader to carry out this obvious adjustment.

No classification of the subgroups of $\mathbf{Q}^n$ for $n \geq 2$ is known (at least to the author). This will cause most of the technical difficulties in our subsequent arguments. The main trick is to reduce the rank of a group by the following procedure:

(4.5) Let A be an abelian group with $t(A) = 0$. Let $0 \neq a \in A$. We denote by $V(a)$ the subgroup of the elements $x \in A$ such that there exist integers m, n with $mx = na$. Then $r(V(a)) = 1$, and thus $r(A/V(a)) = r(A) - 1$. Also $t(A/V(a)) = 0$.

We need the following (probably classical) lemma:

(4.6) LEMMA. *Let A be an abelian group with $t(A)$ finite and $r(A) < \infty$. Let B be a finite abelian group. Then $A \otimes B$ is finite.*

PROOF. It is enough to prove Lemma (4.5) for $t(A) = 0$ and $B = \mathbf{Z}/p^s\mathbf{Z}$. Observe that

$$\mathbf{Z}_{(P)} \otimes \mathbf{Z}/p^s\mathbf{Z} = \begin{cases} 0 & \text{if } p \notin P, \\ \mathbf{Z}/p^s\mathbf{Z} & \text{if } p \in P. \end{cases}$$

This, together with Lemma (4.3), proves Lemma (4.6) for $r(A) = 1$. The general case is now obtained by induction, using the exact sequence

$$V(a) \otimes B \rightarrow A \otimes B \rightarrow A/V(a) \otimes B$$

where $V(a)$ is defined in (4.5) for $0 \neq a \in A$.

Let A be an abelian group and K be an integer ≥ 2 . A Moore space $M(A; k)$ is a simply connected space such that

$$H_i(M(A; k)) \simeq \begin{cases} 0 & \text{if } i \neq k, \\ A & \text{if } i = k. \end{cases}$$

This condition determines the homotopy type of $M(A; k)$. If $f: A \rightarrow B$ is a homomorphism between abelian groups, there is a (not unique) map $F: M(A; k) \rightarrow M(B; k)$ such that $H_k(F) = f$.

(4.7) LEMMA. *Let A be an abelian group with $r(A) = 1$ and $t(A) = 0$. Then, for $k + 1 \leq i$ and $i \neq 2k - 1$, the group $\pi_i(M(A; k))$ is finite and its order divides the order of $\pi_i(S^k)$.*

PROOF. Suppose that $A \simeq \mathbf{Z}_{(P)}$ for some set of primes P . The space $M(A; k)$ is then the localization of S^k at P [Su, §2] and $\pi_i(M(A; k)) \simeq \pi_i(S^k) \otimes \mathbf{Z}_{(P)}$. This proves Lemma (4.7) in this particular situation. The general case follows from Lemma 4.3 which implies that $M(A; k)$ has the homotopy type of $\varinjlim M(\mathbf{Z}_{(P)}; k)$.

REMARK. The restriction $i \neq 2k - 1$ is necessary only if k is even, since $\pi_i(S^k)$ is finite for k odd and $i \geq k + 1$.

(4.8) PROPOSITION. *Let A be an abelian group with $t(A)$ finite and $r(A) < \infty$. Let k and i be two integers so that $k + 1 \leq i \leq 3k - 3$ and $i \neq 2k - 1$. Then $\pi_i(M(A; k))$ is finite.*

PROOF. *Case 1.* $t(A) = 0$. The proof goes by induction on $r(A)$. Lemma 4.7 gives the result for $r(A) = 1$. Let us assume that Proposition (4.8) is true for any abelian A' with $t(A') = 0$ and $r(A') \leq r - 1$. Let A be an abelian group with $t(A) = 0$ and $r(A) = r$. Let $a \neq 0 \in A$ and construct the subgroup $V(a)$ of (4.5). The epimorphism $A \rightarrow A/V(a)$ gives rise to a homotopy fibration

$$F \rightarrow M(A; k) \rightarrow M(A/V(a); k)$$

whose Serre spectral sequence gives the following information on $H_*(F)$:

$$H_i(F) = \begin{cases} V(a) & \text{if } i = k, \\ V(a) \otimes A/V(a) & \text{if } i = 2k - 1, \\ 0 & \text{if } i \neq k, 2k - 1 \text{ and } i \leq 3k - 3. \end{cases}$$

Let $M(V(a); k) \rightarrow F$ be a map inducing an isomorphism on $H_k(\quad)$, and let G be its homotopy theoretic fiber. The Serre spectral sequence now gives:

$$H_i(G) = \begin{cases} V(a) \otimes A/V(a) & \text{if } i = 2k - 2, \\ 0 & \text{if } i \neq 2k - 2 \text{ and } i \leq 3k - 4. \end{cases}$$

Thus G has the $(3k - 4)$ -type of $M(V(a) \otimes A/V(a); 2k - 1)$. Observe that $r(V(a) \otimes A/V(a)) = r - 1$. By the induction hypothesis, one has that $\pi_i(G)$ is finite for $i \neq 2k - 1$ and $i \leq 3k - 4$. By applying the inductive hypothesis again, the result for $M(A; k)$ follows using the homotopy exact sequence of the above two fibrations.

General case. One considers the fibration

$$L \rightarrow M(A) \rightarrow M(A/t(A)).$$

The fibre L is simply connected since $k \geq 2$. The Serre spectral sequence of this fibration together with Lemma (4.6) implies that $H_i(L)$ are finite for all i . Thus $\pi_i(L)$ are finite for all i by the Serre-Hurewicz theorem [Hu, Theorem 9.1]. Then the general case follows from the case where $t(A) = 0$. Proposition (4.8) is thus proved.

(4.9) PROPOSITION. *Let A be an abelian group with $t(A) = 0$ and $r(A) < \infty$ and let k be an integer ≥ 2 . Suppose that A contains a subgroup isomorphic to $\mathbb{Z}[1/k!]^{r(A)}$. Then $\pi_i(M(A; k)) = 0$ for $k + 1 \leq i \leq 3k - 3$ and $i \neq 2k - 1$.*

PROOF. Also by induction on $r(A)$. If $r(A) = 1$, the group A is isomorphic to a direct limit of groups isomorphic to $\mathbb{Z}_{(P)}$ where P does not contain any prime $\leq k$. (See Lemma(4.3).) Then $M(A; k) \simeq \varinjlim M(\mathbb{Z}_{(P)}; k)$. By the localization theory [Su, §2] one has $\pi_i(M(\mathbb{Z}_{(P)}; k) \simeq \pi_i(S^k) \otimes \mathbb{Z}_{(P)}$. By [Hu, Corollary 9.3 and 13.3] one has that $\pi_i(S^k) \otimes \mathbb{Z}_{(P)} = 0$ for $k + 1 \leq i \leq 3k - 3$ and $i \neq 2k - 1$. This proves Proposition (4.9) for $r(A) = 1$. The induction argument goes now exactly as in the proof of Proposition (4.8).

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