

## EASY $S$ AND $L$ GROUPS

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**ABSTRACT.** We give a simple proof that the existence of strong  $S$  or  $L$  spaces implies the existence of strong  $S$  or  $L$  groups; in fact the algebraic structure can be varied quite a bit. We also construct, under CH,  $S$  and  $L$  groups whose squares are neither  $S$  nor  $L$ .

**0. Introduction and preliminaries.** Some definitions. An  $S$ -space is regular and hereditarily separable but not hereditarily Lindelöf; an  $L$ -space is regular and hereditarily Lindelöf but not hereditarily separable. We say that a space has countable spread if it has no uncountable discrete subspace. For a topological property  $P$ , a space  $X$  is said to be strong  $P$  iff every finite product of copies of  $X$  has property  $P$ . A canonical space is a subspace of  $2^{\omega_1}$  and if  $X = \{f_\alpha: \alpha < \omega_1\}$  is canonical then its dual is the space  $X^D = \{g_\beta: \beta < \omega_1\}$  where  $g_\beta(\alpha) = f_\alpha(\beta)$ . The dual is, of course, not unique and depends upon the enumeration of  $X$ .

It is well known that  $L$ -spaces exist iff there is a canonical  $L$ -space, and hence a canonical  $L$ -space of cardinality  $\aleph_1$ . The same argument works for strong  $L$ -spaces and strong  $S$ -spaces. It is easy to show that strong  $L$ -spaces exist iff there are strong  $S$ -spaces. One proof is based on the observation that a canonical space of cardinality  $\aleph_1$  is a strong  $L$ -space iff its dual is strong  $S$ . Another proof is given by a theorem due to Zenor. Let  $H$  be separable Hilbert space and let  $C(X, H)$  be the space of all continuous functions from  $X$  into  $H$ , equipped with the topology of pointwise convergence. Combining a result of Zenor [Z] with standard techniques from function spaces (e.g. [E, Proposition 26.9]) one can see that  $X$  is a strong  $S$ -space iff  $C(X, H)$  is a strong  $L$ -space, and that  $X$  is a strong  $L$ -space iff  $C(X, H)$  is a strong  $S$ -space.

Canonical strong  $S$  and  $L$  spaces exist under CH (modify the construction of [HJ<sub>2</sub>]), in a forcing extension similar to that of [HJ<sub>1</sub>] (the cardinality is  $2^{2^{\aleph_1}}$ ), in models obtained by adding one Cohen or random real [R] and hence in a direct iterated property  $K$  extension (by [KT] such an extension does not destroy  $S$  or  $L$  spaces in inner models extending the ground model and by an observation of Kunen there is an inner model adding one Cohen real to the ground model). On the other hand, strong  $S$  and  $L$  spaces do not exist under  $\text{MA} + \neg\text{CH}$  [K].

(1) If there is a strong  $S$  space, then there is a strong  $S$  group, i.e. a topological group which is a strong  $S$ -space.

(2) If there is a strong  $L$ -space, then there is a strong  $L$ -group.

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Those assertions can be verified by applying Zenor's results twice, e.g., if  $X$  is strong  $S$ , then  $C(X, H)$  is strong  $L$  so that  $C(C(X, H), H)$  is a strong  $S$ -space, and the latter function space is clearly a topological group. In this paper we give a direct and simple proof of these two corollaries. Our proofs have the advantage that they adapt easily to give other structures (e.g., nonabelian groups, topological rings, Boolean rings) which are strong  $S$  or  $L$  space. Since, as noted above, the existence of strong  $S$  or  $L$  spaces is known to be consistent in many models, this gives two proofs of the consistency of strong  $S$  and  $L$  groups. There is yet a third proof of the consistency of strong  $S$ -groups, due to Juhász and Hajnal, obtained by varying the constructions of  $[HJ_1]$  and  $[HJ_2]$ . We vary this construction to show that although a topological group is closely allied to its product spaces, an  $S$  or  $L$  group need not be strong, e.g. under  $CH$  there are  $S$  and  $L$ -groups whose squares are not, respectively,  $S$  or  $L$ .

There are several questions that remain open. For example, we do not know if there is a topological field which is an  $S$  or  $L$  space and we do not know whether  $MA + \neg CH$  implies that no  $S$  or  $L$  groups can exist. (However, we give an example of the sort of  $S$  or  $L$  group easily ruled out by  $MA + \neg CH$ .)

**1. Strong  $S$  and  $L$  groups.** If  $X^*$  is a topological group, and  $X \subset X^*$ , we prove that  $\text{cl}(X)$ , the closure of  $X$  under the group operation of  $X^*$ , preserves the following properties: strong countable spread, strong hereditary separability, strong hereditary Lindelöfness. This will follow from the following.

*Observation.* Each of the properties countable spread, hereditary separability, and hereditary Lindelöfness is preserved by continuous images and countable union.

(Countable union here does not have to mean disjoint union;  $X$  is the countable union of  $\{Y_n: n < \omega\}$  iff  $X = \bigcup_{n < \omega} Y_n$ , each  $Y_n$  a subspace of  $X$ .)

All we have to show, then, is that each  $[\text{cl}(X)]^n$  is the countable union of continuous images of suitable  $X^k$ 's; since  $[\text{cl}(X)]^n$  contains a copy of  $X$ , it is not hereditarily Lindelöf if  $X$  is not, and not hereditarily separable if  $X$  is not. So we will have shown that  $X$  strong  $S \Rightarrow \text{cl}(X)$  is,  $X$  strong  $L \Rightarrow \text{cl}(X)$  is.

First we deal with  $\text{cl}(X)$ . Let  $Y_n = \{\prod_n x_i^{m_i}: x_i \in X, m_i \in \{-1, 1\}\}$ . Thus  $\text{cl}(X) = \bigcup_{n < \omega} Y_n$ . Map  $X^n$  onto  $Y_n$  by  $\varphi \langle x_0 k, \dots, x_{n-1} \rangle = \prod_{i < n} x_i^{m_i}$ .  $\varphi$  is continuous.

Now we deal with  $[\text{cl}(X)]^n$ . Given  $t = \langle k_0, \dots, k_{n-1} \rangle$  an increasing sequence of integers, we define  $\varphi_t: X^{k_{n-1}+1} \rightarrow [\text{cl}(X)]^n$  by

$$\varphi_t \langle x_0, \dots, x_{k_{n-1}} \rangle = \left\langle \prod_{j \leq k_0} x_j^{m_j}, \dots, \prod_{k_{n-2} < j \leq k_{n-1}} x_j^{m_j} \right\rangle.$$

Again,  $\varphi_t$  is continuous. Let  $Z_t$  be the image of  $\varphi_t$ . Then  $[\text{cl}(X)]^n$  is the countable union of the  $Z_t$ 's, and we are done.

This construction has many variations. For example, if  $X$  is a canonical space and  $G$  a group,  $f \in X$ ,  $a \in G$ , we define  $af(\alpha) = a$  if  $f(\alpha) = 1$ ;  $af(\alpha) = 0$  otherwise. Then identifying the identity of  $G$  with 0,  $GX$  is defined as the closure of  $\{af: a \in G, f \in X\}$  under the coordinatewise group operation, where the topology is as

a subspace of  $G^{\omega_1}$ ,  $G$  discrete. Then if  $G$  is countable nonabelian, and  $X$  is a strong  $S$  or  $L$  group,  $GX$  is a strong  $S$  or  $L$  nonabelian group. On the other hand, if  $G$  is the free group on the generators  $\{a_i: i < \omega\}$  and  $X$  is canonical,  $X^n$  is isomorphic to a subspace of  $\prod_{i < n} a_i X$ ; hence under  $\text{MA} + \neg\text{CH}$ ,  $GX$  is not an  $S$  or  $L$  subspace of  $G^{\omega_1}$ .

As another variation, we can get a ring whose additive group is generated by multiplicative units by considering our strong  $S$  or  $L$  group as a subspace of  $\{-1, 1\}^{\omega_1}$  under coordinatewise multiplication and then closing off under addition.

In another variation, adding the constant function 1 to our original  $\text{cl}(X)$  and closing under difference, addition, and multiplication gets us a strong  $S$  or  $L$  Boolean ring.

Finally, we note that if  $X$  is canonical with strong countable spread, and is not the union of an  $S$  or  $L$  space, so is  $\text{cl}(X)$  (such an  $X$  exists under  $\text{CH}$  by adapting the proof of  $[\mathbf{R}_1]$ , noting that the Kunen line  $[\mathbf{JKR}]$  may be made strong by an adaptation similar to those of the next section, and a strong  $L$  space may be used as the vertical space of the construction).

**2. Groups whose squares are not  $S$  or  $L$ .** We construct  $S$  and  $L$  groups whose squares are not  $S$  or  $L$  by closely imitating the remarkably flexible construction of Juhász and Hajnal  $[\mathbf{HJ}_2]$ , assuming  $\text{CH}$ . The reader familiar with their similar constructions from  $V = L$  (via Silver's property  $W$ ) and with forcing  $[\mathbf{HJ}_1]$  will easily see that our adaptations work in those contexts as well.

A canonical set  $X$  is said to be hereditarily finally dense (HFD) iff for every countable  $Y \subset X$  there is some  $\alpha_Y$  so that if  $\varepsilon$  is a finite function from  $\omega_1 - \alpha_Y$  into 2, then some  $f \in Y$  extends  $\varepsilon$ . If  $X, X'$  are canonical,  $X \equiv X' \pmod{\text{countable}}$  iff  $\exists \psi: X \rightarrow X'$  a set isomorphism so that  $\forall f \in X \exists \alpha_f < \omega_1 (f(\beta) = \psi(f)(\beta) \text{ for all } \beta < \alpha_f)$ . HFD's have the nice property that if two sets are equivalent mod countable, and one is an HFD, so is the other. Hence if  $X$  is an uncountable HFD, a little fiddling gives you a non-Lindelöf subspace, whereas a  $\Delta$ -system argument shows that every HFD is hereditarily separable. So, if there is an HFD, then there is an  $S$  space.

The dual of the HFD concept is more awkward to state. Since we follow the Juhász and Hajnal constructions so closely, we refer the reader to  $[\mathbf{HJ}_2]$  for the  $L$  space case, and only indicate briefly our adaptation here.

How did Juhász and Hajnal construct an HFD from  $\text{CH}$ ? First they named the functions,  $\{f_\alpha: \alpha < \omega_1\}$ , although of course they did not yet know what they are. Then, using  $\text{CH}$ , they enumerated all countable infinite subsets of these names as  $\{Y_\alpha: \alpha < \omega_1\}$ . Then, for each  $\alpha < \omega_1$ ,  $A_\alpha = \{\varepsilon_i: i < \omega\}$  enumerated all finite functions from  $\alpha$  into 2, and  $B_\alpha = \{\beta < \alpha: f_\gamma \in Y_\beta \Rightarrow \gamma < \alpha\}$ . For  $\varepsilon \in A_\alpha$ ,  $\beta \in B_\alpha$ , they let  $Y_{\beta, \varepsilon} = \{f \in Y_\beta: f \supset \varepsilon\}$ . Finally,  $C = \{Y: Y \text{ infinite and } Y = Y_{\beta, \varepsilon} \text{ for some } \beta \in B_\alpha, \varepsilon \in A_\alpha\}$ . The  $f_\gamma$ 's were simultaneously built up by induction, where at stage  $\alpha$  each  $f_\gamma(\alpha)$  was determined, for  $\gamma < \alpha$ . The induction hypothesis at stage  $\alpha$  said that if  $Y \in C_\beta$ ,  $\beta < \alpha$ , then both  $\{f \in Y: f \supset \langle \beta, 0 \rangle\}$  and  $\{f \in Y: f \supset \langle \beta, 1 \rangle\}$  were infinite. The construction was continued at stage  $\alpha$  by enumerating  $C_\alpha$  as  $\langle Z_i: i < \omega \rangle$  where each  $Y \in C_\alpha$  appeared as infinitely many  $Z_i$ 's; then,

one by one, the  $Z_i$ 's were picked off so the induction hypothesis would be satisfied. Thus an HFD was born.

To get an HFD group they added to the list of names  $\{\Sigma_{i < n} \alpha_i: \alpha_i < \omega_1\}$  where addition is again coordinatewise mod 2. Then  $\{Y_\alpha: \alpha < \omega_1\}$  enumerates all countable infinite sets of these names, etc. The induction hypothesis is the same, and addition will not get us into trouble—at any point in the  $\alpha$ th stage of the construction only finitely much has been determined, and so the  $Z_i$  we are looking at contains at least one sum mentioning functions whose  $\alpha$ th coordinate is not yet decided.

Define  $X$  to be a strong HFD if for any infinite  $Y$  a subset of any  $X^n$  there is an  $\alpha_Y$  so that if  $\varepsilon_i$  is a finite function from  $\omega_1 - A_Y$  into 2, for all  $i < n$ , then there is  $\langle f_i: i < n \rangle \in Y$  with  $f_i \supset \varepsilon_i$ , for all  $i < n$ . Then similar constructions give a strong HFD, where now we add the names  $\{\langle f_\alpha: i < n \rangle: n < \omega, \alpha_i < \omega_1\}$ . This construction is a slight strengthening of the one implicit in Galvin's construction from CH of a ccc space whose square is not ccc. (Note that implicit in  $[R_2]$  is the proof that a strong HFD gives the underlying graph of Galvin's space.)

How do we get an HFD group whose square is not  $S$ ? We proceed as in the second variation with the following twist. Let  $a_\alpha = \langle f_{\beta_\alpha}, f_{\gamma_\alpha} \rangle$  where all indices are distinct, and let  $u_\alpha = \langle \langle \alpha, 0 \rangle, \langle \alpha, 0 \rangle \rangle$  (i.e.  $\langle f, g \rangle \in u_\alpha$  iff  $f(\alpha) = g(\alpha) = 0$ ). Before beginning our construction, we insist that

- (1)  $a_\alpha \in u_\alpha$ ,
- (2)  $\alpha \neq \beta \Rightarrow a_\alpha \notin u_\beta$ .

By the remark on equivalence mod countable, it suffices to insist that  $\alpha \neq \beta$ ,  $\beta_\alpha, \gamma_\alpha < \beta \Rightarrow a_\alpha \notin u_\beta$ ; and by the same remark, (1) is harmless. Similarly, by playing with a suitable subset, we can ensure an HFD group is actually  $S$ .

In the actual construction, we must simultaneously assign values to  $f_{\beta_\alpha}, f_{\gamma_\alpha}$  and  $f_{\beta_\alpha} + f_{\gamma_\alpha}$ . But this does no harm to the inductive hypothesis. Since either  $f_{\beta_\alpha}(\beta) = f_{\gamma_\alpha}(\beta) = 1$  or  $f_{\beta_\alpha}(\beta) \neq f_{\gamma_\alpha}(\beta)$  is possible,  $\alpha \neq \beta$ , we are free to let whichever of  $f_{\beta_\alpha}, f_{\gamma_\alpha}$  or  $f_{\beta_\alpha} + f_{\gamma_\alpha} = 0$  or 1 on  $\beta$ , as needed.

We remark that similar tricks give an HFD group  $X$  such that  $X^n$  is  $S$  but  $X^{n+1}$  is not, for each  $n < \omega$ .

Finally, to construct an  $L$  group from CH whose square is not  $L$ , we imitate the construction of  $[HJ_2]$  dual to the HFD construction, using the same a priori conditions as above, where  $u_\alpha$  is defined by  $u_\alpha = \langle \langle \alpha, 0 \rangle, \langle \alpha, 1 \rangle \rangle$ . Again, we can modify the construction to get  $X$  an  $L$  group,  $X^n$  is  $L$ ,  $X^{n+1}$  is not  $L$ ,  $\forall n < \omega$ .

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