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DIFFERENTIABLE DECOMPOSITIONS OF MANIFOLDS INTO TOTALLY C^∞-PATH DISCONNECTED SUBSETS

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ABSTRACT. For C^∞-manifolds M, N, the set of all C^s-maps $M \to N$ with totally C^∞-path disconnected fibers is shown to be dense in the set of all C^s-maps $M \to N$, if $\dim N > 0$.

1. Introduction. R. D. Anderson [1] in 1950, and Lewis and Walsh [3] more recently, have given a continuous decomposition of the plane into pseudo-arcs. This gives a nontrivial example of a continuous map of the plane to itself with totally path disconnected fibers (= point inverses). Such maps were called *a-light* by Ungar [7]. One can consider a differentiable analogue of such maps and seek a nontrivial C^∞-decomposition of a C^∞-manifold into totally C^∞-path disconnected subsets. But, if by a C^∞-decomposition of M we mean a decomposition that can be induced by a C^∞-map of M to some C^∞-manifold N, then, as a consequence of Sard's theorem [6], "most" elements of the decomposition will be C^∞-submanifolds of dimension $\dim M - \dim N$, and thus will not be totally C^∞-path disconnected if $\dim M > \dim N$. However, if we consider C^s-decompositions, where $s < \infty$, then the situation is different. In this case we show, using results of [4], not only the existence of C^s-decompositions into totally C^∞-path disconnected elements, but that "many" C^s-decompositions are of this type. More explicitly, and more generally, we show that for any integer $s > 0$ and for any C^∞-manifolds M and N with $\dim N > 0$, the set $D_r^s(M, N)$ of all C^s-maps $M \to N$ with totally C^r-path disconnected fibers is dense, relative to the fine C^1-topology, in the set $C^s(M, N)$ of all C^s-maps $M \to N$, if r is sufficiently large.

2. Main results. Define $L: R \to \{0, 1, \dots\}$ by $L(p) = \{\log_2(p)\}$ if $p \geqslant 1$, and 0 otherwise, where $\{x\}$ = least integer $\geqslant x$.

2.1 THEOREM. *If N, M are C^∞-manifolds of dimension $n > 0$ and m respectively and s is a positive integer then $\overline{D_r^s(M, N)} = C^s(M, N)$, where $^-$ denotes closure relative to the fine C^1-topology and $r = s + 1 + \max\{L(m-1), L(m/n)\}$.*

Since $D_s^s(M, N) = C^s(M, N)$ if $m = 0$ and, by an easy calculation, $\max\{L(m - 1), L(m/n)\} \leqslant m - 1$, if $m > 0$, 2.1 clearly implies:

2.2 COROLLARY. $\overline{D_r^s(M, N)} = \overline{D_\infty^s(M, N)} = C^s(M, N)$ *for* $s = 1, 2, \dots,$ *if* $r \geqslant s + \dim M$ *and* $\dim N > 0$.

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Theorem 2.1 readily follows from the next two propositions involving the sets $E_r^s(M, N) = \{f | f \in C^s(M, N)$, such that for all C^r-paths β in M, with a non-empty, connected, open subset of R as domain, $f\beta \in C^s$ implies β is constant$\}$ since $E_{r_1}^s \subset E_r^s$ if $r > r_1$.

2.3 PROPOSITION. *For* $r > s$, $\overline{D_r^s(M, N)} = C^s(M, N)$ *if* $E_r^s(R^m, R^n) \neq \varnothing \neq E_r^s(R^{m-1}, R)$, *where* $m = \dim M > 0$ *and* $n = \dim N > 0$.

2.4 PROPOSITION. $E_r^s(R^m, R^n) \neq \varnothing$ *if* $r = s + 1 + L(m/n)$ *and* $n > 0$.

To show 2.3 some preliminary results are necessary. Define a subset K of M to be an *r-set* if any C^r-path in K is constant.

2.5 LEMMA. (a) *If* $\{K_i\}$ *is a locally finite family of closed r-sets in M then* $\bigcup_i K_i$ *is also a closed r-set.* (b) *If K is a closed r-set in M and* $f \in C^s(M, N)$ *restricts to a map in* $E_r^s(M - K, N)$ *then* $f \in E_r^s(M, N)$. (c) *If* $r > s$, $E_r^s(R^m, R) \neq \varnothing$, *and x is a point in an open subset U of* R^{m+1}, *then there is an open set V*, $x \in V \subset \bar{V} \subset U$, *such that* $\delta V = \bar{V} - V$ *is an r-set.* (d) *If* $r > s, f \in C^{s+1}(M, R^n)$, $g \in C^\infty(M, [0, 1])$, *and* $h \in E_r^s(M, R^n)$ *then* $(f + gh) \in E_r^s(g^{-1}(0, 1], R^n)$.

PROOF. (a) Since $\{K_i\}$ is locally finite, it suffices to prove the result for two closed r-sets K_1, K_2. For $(\beta\colon U \to K = K_1 \cup K_2) \in C^r$, let $U_1 = \beta^{-1}((K - K_1) \cup (K - K_2))$ and $U_2 =$ interior $(U - U_1)$. Since $(K - K_1) \subset K_2$, $(K - K_2) \subset K_1$, $(K - K_1) \cap (K - K_2) = \varnothing$, and $\beta(U_2) \subset K_1 \cap K_2 \subset K_1$ the assumptions imply that the continuous map β is locally constant on the dense subset $U_1 \cup U_2$ of the connected set U, i.e., β is constant. (b) For $(\beta\colon U \to M) \in C^r$, let $U_1 = \beta^{-1}(M - K)$, $U_2 =$ interior$(U - U_1)$ and proceed as in (a), noting that $f\beta \in C^{s+1}$ implies β is locally constant on U_1. (c) If $E_r^s(R^m, R) \neq \varnothing$ then clearly there is a $g \in E_r^s(R^m, (0, 1))$. Further, if $t \in R - \{0\}$, then the image $K_i(t)$ of the map t_i: $R^m \to R^{m+1}$ given by

$$t_i(x_1, \ldots, x_m) = (x_1, \ldots, x_{i-1}, tg(x_1, \ldots, x_n), x_i, \ldots, x_m), \qquad i = 1, 2, \ldots, m + 1,$$

is a closed r-set in R^{m+1}. Indeed, if β is a C^r-path in $K_i(t)$ then $t_i(d_i\beta) = \beta \in C^r \subset C^{s+1}$, where $d_i\colon R^{m+1} \to R^m$ is deletion of the ith coordinate. Since t_i is clearly in $E_r^s(R^m, R^{m+1})$ and $d_i\beta \in C^r$, it follows that $d_i\beta$, and consequently β, is constant. If $\varepsilon > 0$ then $K = \bigcup_{i=1}^{m+1} (K_i(\varepsilon) \cup K_i(-\varepsilon))$ is a closed r-set in R^{m+1} by (a) and the component V of the origin in $R^{m+1} \to K$ is such that $\delta V \subset K$ and $V \subset (-\varepsilon, \varepsilon)^{m+1}$. This clearly implies (c). (d) If β is a C^r-path in $g^{-1}(0, 1]$ with $(f + gh)\beta = \alpha \in C^{s+1}$, then $h\beta = (\alpha - f\beta)/(g\beta) \in C^{s+1}$ and thus β is constant. This shows 2.5.

For an open subset U of R^m and $f \in C^s(U, R^n)$ define $|f|\colon U - R$ by $|f|(x) =$ the maximum of the absolute value of all the partials of order $\leqslant 1$ at x of the n-component functions of f. Clearly $|f|$ is continuous and if $f = g$ on $K \subset U$ then $|f| = |g|$ on the interior of K.

PROOF OF 2.3. It suffices to show that $\underline{C^{s+1}(M, N)} \subset \overline{E_r^s(M, N)}$ since $E_r^s(M, N) \subset D_r^s(M, N)$ and, by 4.2 of [5], $\overline{C^{s+1}(M, N)} = C^s(M, N)$. Given $f \in C^{s+1}(M, N)$, it is possible, with the aid of 2.5(c), to find C^∞-coordinate systems $\{U_i, h_i\}$, $\{W_i, k_i\}$ in M, N respectively together with an open cover $\{V_i\}$ of M for which $f(U_i) \subset W_i$, $\overline{V}_i \subset U_i$, $\overline{V}_i$ is compact, δV_i is an r-set and $\{\overline{V}_i\}$ is locally finite, $i \in I$. If, for a family $\delta_* = \{\delta_i > 0\}$ of constants and a well ordering of the index set I, there is a family $p_i \in C^s(M, N)$ such that,

(1) $p_i = p_j$ on V_j for $j < i$,

(2) $p_i = f$ on $M - K_i$, where $K_i = \bigcup_{k \leqslant i} V_k$,

(3) $|(p_i)_j - (f)_j| < \delta_j/2$ on $h_j(V_i \cap V_j)$ for $i \leqslant j$, where $(\ )_j = k_j(\ )h_j^{-1}$,

then the unique C^s-map $p: M \to N$ with $p = p_i$ on V_i clearly satisfies $|(p)_i - (f)_i| < \delta_i$ on $h_i(\overline{V}_i)$ by $(3)_{ii}$. Therefore, by 3.6 of [5], p is in the δ_*-neighborhood of f. To show $f \in \overline{E_r^s(M, N)}$, then, it suffices to construct such a family for which $p \in E_r^s(M, N)$. To this end assume that the family $\{p_k\}$ is given for all $k < i$ (take $p_0 = f$, $K_0 = \varnothing$) and define p_i as follows. By (1) and (2) there is a unique C^s-map p_i': $M \to N$ that coincides with p_k and f on V_k and on $M - K_i'$ respectively, where $K_i' = \bigcup_{k<i} V_k$. Let $q_i = (p_i')_i + \varepsilon\alpha_i g_i$: $h_i(U_i) \to k_i(W_i)$, where $\alpha_i \in C^\infty(h_i(U_i), [0, 1])$ is such that $\alpha_i^{-1}(0) = h_i(U_i \cap (M - (K_i - \overline{K}_i')))$ (α_i exists by [2, p. 24]) and where $g_i \in E_r^s(h_i(U_i), R^n)$. Since $p_i' = f$ on $(M - K_i') \supset ((K_i - \overline{K}_i') \cap V_j) = K_{ij}$ it follows that $\theta_{ij} = (k_i^{-1}q_i h_i)_j - (f)_j = k_{ji}[(f)_i + \varepsilon\alpha_i g_i]h_{ij} - k_{ji}(f)_i h_{ij} = \psi_{ij}$ on $h_j(K_{ij})$, where $(\ )_{ij} = (\ )_i(\ )_j^{-1}$. Consequently $|\theta_{ij}| = |\psi_{ij}|$ on $h_j(\overline{K}_{ij})$. Since all of the functions making up ψ_{ij} together with all their partials of order $\leqslant 1$ are continuous and since $\overline{K}_{ij}$ is compact, it is clear, from the form of ψ_{ij}, that the constant $\varepsilon > 0$ can be picked so that $|\psi_{ij}| < \delta_j/2$ on $h_j(\overline{K}_{ij})$. Further, since $\{V_i\}$ is locally finite, the set $J = \{j | i \leqslant j, K_{ij} \neq \varnothing\}$ is finite and so ε can be chosen so that $(4)_j$ $|\theta_{ij}| < \delta_j/2$ on $h_j(K_{ij})$ for all $j \in J$. Finally, take ε smaller, if necessary, in order that $q_i(h_i(V_i))$, and thus $q_i(h_i(U_i))$, is contained in $k_i(W_i)$. The map

$$p_i = \begin{cases} p_i' & \text{on } M - V_i, \\ k_i^{-1}q_i h_i & \text{on } U_i \end{cases}$$

is clearly in $C^s(M, N)$ and satisfies condition (1) since $p_i = p_i'$ on $(M - (K_i - \overline{K}_i')) \supset V_k$. Further, since $M - K_i = (M - V_i) \cap (M - K_i')$, $p_i = p_i' = f$ on $M - K_i$ and condition (2) holds. Since $h_j(V_k \cap V_i \cap V_j) \subset h_j(V_k \cap V_i) \cap h_j(V_k \cap V_j)$, conditions (1) and $(3)_{kj}$ imply that $|(p_i)_j - (f)_j| < \delta_j/2$ on $h_j(V_k \cap V_i \cap V_j)$ and thus also on $h_j(K_i' \cap V_i \cap V_j)$. By definition, $(p_i)_j - (f)_j = \theta_{ij}$ on $K_{ij} \subset U_i$ and, by $(4)_j$, $|(p_i)_j - (f)_j| < \delta_j/2$ on $h_j(K_{ij})$. Since $(K_i' \cap V_i \cap V_j) \cup (K_{ij})$ is dense in $V_i \cap V_j$, condition $(3)_{ij}$ also holds. Finally, if we assume inductively that p_k restricts to a map in $E_r^s(V_k, N)$ for $k < i$, then the same is true for $k = i$. Indeed, since $p_i' = f$ on $(M - K_i') \supset (K_i - \overline{K}_i') = h_i^{-1}(\alpha^{-1}(0, 1])$, 2.5(d) implies that $q_i \in E_r^s(\alpha^{-1}(0, 1], k_i(W_i))$ and consequently, since $p_i = k_i^{-1}q_i h_i$ on $U_i \supset (K_i - \overline{K}_i')$, that p_i restricts to a map in $E_r^s(K_i - \overline{K}_i', N)$. Since $p_i = p_k$ on V_k, p_i also restricts to a map in $E_r^s(K_i' \cap V_i, N)$. Lemma 2.5(b) then implies that p_i restricts to a map in $E_r^s(V_i, N)$ since, by 2.5(a), $[V_i - (K_i' \cap V_i) \cup (K - \overline{K}_i')] \subset (\bigcup_{k<i} \delta V_k) \cap V_i$ is an r-set. Since $p = p_i$ on V_i, $p \in E_r^s(M, N)$ and 2.3 is proved.

In the terminology of §4 of [**4**], the set of (r, s)-maps from R^m to R^n coincides with the set $E_r^s(R^m, R^n)$. Proposition 4.3 of [**4**], showing the existence of (r, s)-maps if $r = s + 1 + L(m/n)$, then gives 2.4 and consequently 2.1.

REMARK. The conditions in Theorem 2.1 subsume $r > s$. If $r \leqslant s$ the rank theorem [**2**, p. 273] readily implies that $D_s^r(M, N) = \varnothing$ if $\dim M > \dim N$. This follows since then for any C^s-map $f\colon M \to N$ there are C^s-coordinate patches centered at p and $f(p)$ respectively relative to which f has the form $f(x_1, \ldots, x_m) = (x_{\cdot}, \ldots, x_k, 0, \ldots, 0)$ where p is a point of M where the rank k of f is maximum and where $m = \dim M$. Since $k \leqslant \dim N < m$ the fiber $f^{-1}(f(p))$ contains the C^s-path $\beta(t) = (0, \ldots, t)$. If further, $s > \dim M - \dim N > 0$ then the C^s-version of Sard's theorem [**6**, p. 47] implies that "almost all" fibers of f support nontrivial C^s-paths.

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