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A HUREWICZ-TYPE THEOREM FOR APPROXIMATE FIBRATIONS

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ABSTRACT. This paper concerns conditions on point inverses which insure that a mapping between locally compact, separable, metric ANR's is an approximate fibration. Roughly a mapping is said to be π_i-movable [respectively, H_i-movable] provided that nearby fibers include isomorphically into mutual neighborhoods on π_i [resp. H_i]. An earlier result along this line is that π_i-movability for all i implies that a mapping is an approximate fibration. The main result here is that for a UV^1 mapping, π_i-movability for $i \leqslant k-1$ plus H_k- and H_{k+1}-movability imply π_k-movability of the mapping. Hence a UV^1 mapping which is H_i-movable for all i is an approximate fibration. Also, if a UV^1 mapping is π_i-movable for $i \leqslant k$ and k is at least as large as the fundamental dimension of any point inverse, then it is an approximate fibration. Finally, a UV^1 mapping $f: M^m \to N^n$ between manifolds is an approximate fibration provided that f is π_i-movable for all $i \leqslant \max\{m-n, \frac{1}{2}(m-1)\}$.

1. Introduction and statement of results. Given a proper surjective mapping $p: E \to B$ between locally compact, separable ANR's, we are interested in conditions on the point inverses which insure that p is an approximate fibration (definition below). Earlier results in this direction involve "UV" conditions on point inverses. The mapping $p: E \to B$ is said to be a k-uv [resp., k-UV] mapping provided that p is proper and surjective and for every b in B and every neighborhood U of $p^{-1}(b)$, there is a neighborhood V of $p^{-1}(b)$ in U such that the inclusion induced map $\tilde{H}_k(V) \to \tilde{H}_k(U)$ [resp. $\pi_k(V, e) \to \pi_k(U, e)$] is zero [for each base point e in V]. The notation uv^k [resp., UV^k] means i-uv [resp., i-UV] for all $i \leqslant k$. In the next section we define properties called π_k-movable and H_k-movable which generalize the "UV" properties by allowing nonzero images. One of the fundamental theorems on UV properties is the following Hurewicz-type theorem [**L2**, Theorem 4.2]. If $p: E \to B$ is a UV^{k-1} mapping and a k-uv mapping where $k \geqslant 2$, then p is a UV^k mapping. The purpose of this note is to prove an analogous theorem for movable mappings.

THEOREM A. *Let $p: E \to B$ be a mapping between locally compact separable ANR's. If p is UV^1, π_i-movable for $i \leqslant k-1$, and H_k- and H_{k+1}-movable where $k \geqslant 2$, then p is π_k-movable.*

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As an application, we give the following improvement of **[CD2**, Theorem 3.7] for UV^1-mappings, and a result on mappings between manifolds which generalizes **[L**, Theorem 5.4].

THEOREM B. *If p: $E \to B$ is a UV^1, π_i-movable map for $i \leqslant k$ and each fiber of p has fundamental dimension $\leqslant k$, then f is an approximate fibration.*

THEOREM C. *Let f: $M^m \to N^n$ be a UV^1 mapping between manifolds. If f is π_i-movable for all $i \leqslant k - 1$ where $k \geqslant \max\{m - n + 1, (m + 1)/2\}$, then f is an approximate fibration.*

We use the following terminology and notation in this paper. If p: $E \to B$ is a mapping, $b \in B$ or $U \subset B$, then the fiber $p^{-1}(b)$ is denoted by F_b and $p^{-1}(U)$ is denoted by $\tilde{U}$. Our usual homology and cohomology groups are singular, with integral coefficient groups. If λ is one of the usual homology, (cohomology) or homotopy functors, then $\check{\lambda}(X)$ denotes the inverse (direct) limit of $\lambda(U)$ as U ranges over the neighborhoods of X. An absolute neighborhood retract for metric spaces is abbreviated to ANR. A manifold is assumed to be connected and boundaryless.

2. Movability and lifting properties. Suppose that p: $E \to B$ is a proper surjective map. We say that p has the *approximate homotopy lifting property* (*AHLP*) *for a space X* if for each commutative diagram

$$\begin{array}{ccc} X \times \{0\} & \xrightarrow{g} & E \\ \cap & & \downarrow p \\ X \times I & \xrightarrow[H]{} & B \end{array}$$

and open cover α of B, there is an extension G: $X \times I \to E$ of g such that $p \circ G$ is α-close to H. We say that such a G is an *α-lift* of H. If p has the AHLP for all spaces X, p is an *approximate fibration*.

In **[CD2]**, it was shown that the AHLP for polyhedra can be detected by a homotopy regularity condition on fibers, which we called k-movability. In this paper, it will be convenient to define movability in a slightly more general setting.

Let Λ_0 be the collection of functors $\{\pi_i, H_j | i, j = 0, 1, 2, \ldots\}$. If $\Lambda \subset \Lambda_0$, we say that p is *Λ-movable* provided that given $b \in B$ and any neighborhood U_0 of F_b, there exist open sets U and V with $F_b \subset V \subset U \subset U_0$ such that for each $F_c \subset V$, each $\lambda \in \Lambda$ (and each base point in F_c if relevant), the inclusion induced map sends $\check{\lambda}(F_c)$ isomorphically onto the image of $\lambda(V)$ in $\lambda(U)$. Given such U, V, we say that $\check{\lambda}(F_c)$ is *realized* as the image of $\lambda(V)$ in $\lambda(U)$.

Thus our earlier terminology, k-movable, is replaced in this paper by Λ-movable where $\Lambda = \{\pi_i | i \leqslant k\}$. Subject to this change the result from **[CD2**, Theorem 3.3] says that if E and B are ANR's and f is Λ-movable for $\Lambda = \{\pi_i | i \leqslant k\}$, then f has the AHLP for polyhedra of dimension $\leqslant k$.

We will need the following technical lemma.

LEMMA 2.1. *Let Λ_1 and Λ_2 be subsets of Λ_0. If p: $E \to B$ is Λ_1-movable and Λ_2-movable, then p is $(\Lambda_1 \cup \Lambda_2)$-movable. Furthermore, if B is a manifold, the open sets U and V may be chosen to be preimages of contractible sets.*

PROOF. Given $b \in B$ and $U_0 \supset F_b$, choose open sets $U_1, V_1, U_2, V_2, U_3, V_3$ with $F_b \subset V_3 \subset U_3 \subset V_2 \subset U_2 \subset V_1 \subset U_1 \subset U_0$ such that for every $F_c \subset V_3$, $\lambda \in \Lambda_1$, $\check{\lambda}(F_c)$ is realized as the image of $\lambda(V_3)$ in $\lambda(U_3)$ and as the image of $\lambda(V_1)$ in $\lambda(U_1)$ and such that $\check{\mu}(F_c)$ is realized as the image of $\mu(V_2)$ in $\mu(U_2)$ for every $\mu \in \Lambda_2$. It is easy to check that if $U = U_2$ and $V = V_3$, $\check{\lambda}(F_c)$ is realized as the image of $\lambda(V)$ in $\lambda(U)$ for each $\lambda \in \Lambda_1 \cup \Lambda_2$.

For the second conclusion, given $b \in B$ and $U_0 \supset F_b$ choose open sets $V \subset V_2 \subset U_2 \subset U \subset V_1 \subset U_1$ such that $F_b \subset V$, $U_1 \subset U_0$, $\check{\lambda}(F_c)$ is realized as the image of $\lambda(V_i)$ in $\lambda(U_i)$ for each $F_c \subset V_i$ $(i = 1, 2)$, and U and V are preimages of contractible sets.

In the next section, it will be convenient to assume that E and B are Q-manifolds. A natural device is to replace p by the map $p \times 1_Q$: $E \times Q \to B \times Q$ and appeal to Edwards' Theorem **[E]**. The reader can easily provide a proof for the following lemma.

LEMMA 2.2. *If p: $E \to B$ is a proper map between ANR's, then for each of the properties P_i below, p has P_i if and only if $p \times 1_Q$ has P_i.*

P_1: *Being an approximate fibration.*

P_2: *Being λ-movable for some $\lambda \in \Lambda_0$.*

P_3: *Having the AHLP for a space X.*

LEMMA 2.3. *Suppose that p: $E \to B$ is a map between ANR's and that p has the AHLP for polyhedra of dimension $\leqslant q$. If $V \subset U$ is a pair of open sets in Y, then p_*: $\pi_i(\tilde{U}, \tilde{V}) \to \pi_i(U, V)$ is an isomorphism for $i \leqslant q$ and is epic for $i = q + 1$.*

The proof is a variation of a standard argument; see for example **[S**, Theorem 7.2.8], **[L1**, Corollary 2.4] and **[L2**, Lemma 1.2]. It uses **[CD2**, Lemma 1.2].

3. Proof of Theorem A. By Lemma 2.1 we may assume that both E and B are Q-manifolds. Thus, each point in B has arbitrarily small contractible open neighborhoods.

Let us say that p has property *i-DUV* provided that for each $b \in B$ and each neighborhood U_0 of F_b, there are neighborhoods $V \subset U$ of F_b in U_0 such that given any fiber F_c in V and any neighborhood W_0 of F_c in V, there are neighborhoods $X \subset W$ of F_c in W_0 such that the inclusion induced map $\nu_\#$: $\pi_i(V, X) \to \pi_i(U, W)$ is the zero homomorphism for each base point. By Lemma 3.1 of **[CD2]**, p has property *i-DUV* for $i \leqslant k - 1$. We wish to prove *k-DUV* and $(k + 1)$-*DUV*.

Given $b \in B$ and a neighborhood U_0 of F_b, apply the hypotheses and Lemma 2.1 to choose $V \subset U$ satisfying the following properties.

(i) $\check{\pi}_{k-1}F_c$ is realized as the image of $\pi_{k-1}V$ in $\pi_{k-1}U$ for each $F_c \subset V$,

(ii) $\check{H}_iF_c$ is realized as the image of H_iV in H_iU for each $F_c \subset V$ and $i = k, k + 1$, and

(iii) U and V are the preimages of contractible neighborhoods of b. Given a fiber $F_c \subset V$ and a neighborhood W_0 of F_c in V, apply the hypotheses and the above lemmas again to choose $X \subset W$ satisfying the following properties:

(iv) $\check{\pi}_{k-1}F_c$ is realized as the image of $\pi_{k-1}X$ in $\pi_{k-1}U$,

(v) $\check{H}_iF_c$ is realized as the image H_iX in H_iW, for $i = k, k+1$, and

(vi) X and W are preimages of contractible neighborhoods of b.

Consider the following commutative diagram.

$$\begin{tikzcd}
 & & & & \pi_k(V,X) \arrow[r] \arrow[d] & \pi_{k-1}X \arrow[d, "\Phi_\#"] & \\
 & & & \pi_k V \arrow[r] \arrow[dll] & \pi_k(V,W) \arrow[r] \arrow[d] & \pi_{k-1}W \arrow[r, "\chi_\#"'] & \pi_{k-1}V \\
 & H_kV \arrow[d, "\Psi_\#"] & & & \pi_k(U,W) \arrow[dll, "h"] & & \\
H_kW \arrow[r, "\Psi_\#\chi_\#"] & H_kU \arrow[r] & H_k(U,W) & & & &
\end{tikzcd}$$

The horizontal rows are portions of exact sequences of pairs, the vertical arrows are inclusion-induced and the diagonal arrows are Hurewicz homomorphisms. By **[L2**, Lemma 5.1], (iii), and (vi), U and W are simply connected. Also by (iii) and (vi) and Lemma 2.3, $\pi_i(U, W) = 0$ for $i \leqslant k-1$. Thus, by **[S**, p. 397] h is an isomorphism. By (i) and (iv) $\chi_\#|\text{im } \Phi_\#$ is monic, and by (ii) and (v) im $\Psi_*\chi_* =$ im Ψ_*. It is now an easy "diagram chasing" argument to show that $\nu_\#$ is zero and, hence, we have k-DUV.

Consider next the following similar diagram.

$$\begin{tikzcd}
 & & & \pi_{k+1}(V,X) \arrow[d] & & \\
 & & & H_{k+1}(V,X) \arrow[r] \arrow[d] & H_k(X) \arrow[d, "\Phi_*"] & \\
 & & H_{k+1}(V) \arrow[r] \arrow[d, "\Psi_*"] & H_{k+1}(V,W) \arrow[r] \arrow[d] & H_k(W) \arrow[r, "\chi_*"'] & H_k(V) \\
 & H_{k+1}(W) \arrow[r, "\Psi_*\chi_*"] & H_{k+1}(U) \arrow[r] & H_{k+1}(U,W) & & \\
 & & & \pi_{k+1}(U,W) \arrow[u, "h"'] & &
\end{tikzcd}$$

Since the proof of Theorem 3.3 of **[CD2]** really uses only i-DUV (rather than i-movability as stated), we see that p has AHLP for I^i, $i < i < k$. As above this implies that $\pi_k(U, W) = 0$ and h is an isomorphism. Also $\chi_*|\text{im } \Phi_*$ is monic and im $\Psi_* =$ im $\Psi_*\chi_*$ again. Hence $\nu_\#$ is zero and $(k+1)$-DUV results.

We now apply Theorem 3.3 of **[CD2]** again to see that p has AHLP for I^i, $i \leqslant k + 1$. Hence p is a π_k-movable map by Proposition 3.5 of **[CD2]**, so the proof is finished.

COROLLARY. *Let p: $E \to B$ be a mapping between locally compact, separable metric ANR's. If p is UV^1 and H_i-movable for all i, then p is an approximate fibration.*

When Theorem A is compared to Lacher's Hurewicz-type theorem for UV properties **[L2**, Theorem 4.2], a discrepancy in the analogy is noticeable. There is no hypothesis in Lacher's theorem coresponding to the H_{k+1}-movable hypothesis in Theorem A. The reason for this difference is explained in the remark on page 51 of **[CD2]**. The extra hypothesis is necessary as the following example shows.

Let f: $S^3 \to S^2$ be the Hopf fibration (a generator of $\pi_3(S^2) \cong Z$), and let K be the complex obtained by attaching a 4-cell to the mapping cylinder M_f along S^3 by the identity. It follows that

$$\begin{aligned} &\tilde{H}_i(K) \cong 0, \qquad i \leqslant 1, \\ &H_2(K) \cong Z \cong \pi_2(K), \\ &H_3(K) \cong 0 \cong \pi_3(K), \quad \text{and} \\ &H_4(K) \cong Z. \end{aligned}$$

Let α: $S^2 \to K$ be a generator of $\pi_2(K)$ and let $M_\alpha = (S^2 \times I \cup K)/\{(x, 1) = f(x)\}$ be the mapping cylinder of α. Define p: $M_\alpha \to I$ by a $p[(x, t)] = t, p(K) = 1$. Then p is a π_2-movable map which is H_3-movable and 1-UV, but p is not π_3-movable, since $\pi_3(K) = 0$, $\pi_3(S^2) \neq 0$. Thus we cannot remove the assumption that p be H_{k+1}-movable in Theorem A.

4. Proofs of the applications.

PROOF OF THEOREM B. Since $Fd(F_b) \leqslant k$, $\check{H}^i(F_b) \cong 0$ for $i \geqslant k + 1$ and each $b \in B$. It follows from **[L2**, Theorem 3.1] that p is an i-$uv(Z)$ map for $i \geqslant k + 1$. Hence p is H_i-movable for $i \geqslant k + 1$. By Theorem A, p is π_i-movable for all i, so p is an approximate fibration by **[CD2**, Corollary 3.4].

LEMMA 4.1. *If f: $M^m \to N^n$ is a UV^1, $\{\pi_i | i \leqslant k - 1\}$-movable mapping between manifolds, then for each $y \in N$, $\check{H}^j(F_y) = 0$ for $j \geqslant \max\{m - k + 1, m - n + 1\}$.*

PROOF. For $n = 0, 1$, the result is contained in **[LM**, Theorem 1.3]. For $n \geqslant 2$, take $y \in N$ and $j \geqslant \max\{m - k + 1, m - n + 1\}$. If U is a Euclidean neighborhood of y, then U is simply connected and $\pi_{m-j}(U, U - y) = 0$. By Lemma 2.3, $\pi_{m-j}(\tilde{U}, \tilde{U} - F_y) = 0$. The relative Hurewicz theorem **[S**, p. 397] yields $H_{m-j}(\tilde{U}, \tilde{U} - F_y) = 0$. Since U is simply connected and f is UV^1, $\tilde{U}$ is simply connected **[L2**, Lemma 5.1] and thus orientable **[S**, p. 294]. Therefore duality **[S**, p. 296] can be applied to give $\check{H}^j(F_y) = 0$.

PROOF OF THEOREM C. Let $y \in N$. By the above lemma $\check{H}^j(F_y) = 0$ for all $j \geqslant \max\{m - k + 1, m - n + 1\}$. Since $k \geqslant m - n + 1$ and $k \geqslant (m + 1)/2$, $\max\{m - k + 1, m - n + 1\} \leqslant k$. Hence $\check{H}^j(F_y) = 0$ for all $j \geqslant k$. By **[L**, Theorem 3.1], f is a j-uv map for all $j \geqslant k$; and by Theorem A, f is $\{\pi_i\}_{i=1}^{\infty}$-movable. Hence f is an approximate fibration **[CD2**, Corollary 3.4].

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