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PROOF OF A THEOREM OF BURKE AND HODEL ON THE CARDINALITY OF TOPOLOGICAL SPACES[1]

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ABSTRACT. Techniques of Pol are used to give a direct proof of Burke and Hodel's inequality $|X| \leqslant 2^{\Delta(X) \cdot \mathrm{psw}(X)}$, where $\Delta(X)$ is the discreteness character of the T_1 space X and $\mathrm{psw}(X)$ is the point separating weight of X.

A cover $\mathcal{G}$ of a topological space X is *separating* if for each $x, y \in X$ with $x \neq y$ there is $G \in \mathcal{G}$ with $x \in G$ and $y \notin G$. For a T_1 space X the *point separating weight* $\mathrm{psw}(X)$ of X is $\omega \cdot \mathfrak{n}$, where $\mathfrak{n}$ is the smallest cardinal such that X has a separating open cover $\mathcal{G}$ with $\mathrm{ord}(x, \mathcal{G}) \leqslant \mathfrak{n}$ for all $x \in X$, and the *discreteness character* $\Delta(X)$ of X is $\omega \cdot \mathfrak{n}$, where $\mathfrak{n} = \sup\{|D|: D$ is a closed discrete subset of $X\}$ (see **[BH]**). In **[BH**, 4.4] Burke and Hodel use a version, due to Burke **[BH**, 4.1], of the Erdös-Rado Δ-system lemma to prove the theorem below. Here we give a direct proof suggested by techniques of Pol **[$\mathbf{P_1}$]**. (These in turn derive from ideas of Ponomarev **[$\mathbf{P_2}$]** and Šapirovskiĭ **[S]**; see also **[E**, 3.12.10] and **[BH**, 4.6].)

THEOREM (BURKE AND HODEL). *If X is T_1, then* $|X| \leqslant 2^{\Delta(X) \cdot \mathrm{psw}(X)}$.

PROOF. Let $\mathfrak{m} = \Delta(X) \cdot \mathrm{psw}(X)$, let $\mathcal{G}$ be a separating open cover of X with $\mathrm{ord}(x, \mathcal{G}) \leqslant \mathfrak{m}$ for all $x \in X$, and for each $x \in X$ let $\mathcal{G}_x = \{G \in \mathcal{G}: x \in G\}$. Construct a sequence $(Y_\xi)_{\xi < \mathfrak{m}^+}$ such that:

(1) For each $\xi < \mathfrak{m}^+$, $Y_\xi \subset X$ and $|Y_\xi| \leqslant 2^{\mathfrak{m}}$.

(2) If $\eta \leqslant \xi < \mathfrak{m}^+$, then $Y_\eta \subset Y_\xi$.

(3) If $0 < \xi < \mathfrak{m}^+$ and $\mathfrak{U} \subset \bigcup \{\mathcal{G}_x: x \in \bigcup_{\alpha<\xi} Y_\alpha\}$ with $|\mathfrak{U}| \leqslant \mathfrak{m}$ and $X - \bigcup \mathfrak{U} \neq \varnothing$, then $Y_\xi - \bigcup \mathfrak{U} \neq \varnothing$.

(Set $Y_0 = \varnothing$. If $0 < \xi < \mathfrak{m}^+$ and Y_α is already defined for all $\alpha < \xi$, let E be the set obtained by choosing a point from each nonempty member of $\{X - \bigcup \mathfrak{U}: \mathfrak{U} \subset \bigcup \{\mathcal{G}_x: x \in \bigcup_{\alpha<\xi} Y_\alpha\}, |\mathfrak{U}| \leqslant \mathfrak{m}\}$ and set $Y_\xi = E \cup (\bigcup_{\alpha<\xi} Y_\alpha)$.)

Let $Y = \bigcup_{\xi<\mathfrak{m}^+} Y_\xi$. It suffices to show that $Y = X$. Suppose $p \in X - Y$. For each $x \in Z = X - \{p\}$ there is $G_x \in \mathcal{G}_x$ with $p \notin G_x$. Let $\mathcal{V} = \{G_x: x \in Z\}$ and, by Zorn's lemma, choose $D \subset Z$ such that $D \cap \mathrm{st}(x, \mathcal{V}) = \{x\}$ for all $x \in D$ and $Z \subset \bigcup_{x \in D} \mathrm{st}(x, \mathcal{V})$. Let $\mathfrak{U} = \{U \in \mathcal{V}: U \cap D \neq \varnothing, U \cap Y \neq \varnothing\}$. Note that

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D is closed discrete in $Z = \bigcup \{X - G: G \in \mathcal{G}_p\}$, so $|D| < \mathfrak{m}$ and hence $|\mathcal{U}| < \mathfrak{m}$. Then there is $\beta < \mathfrak{m}^+$ with $U \cap Y_\beta \neq \varnothing$ for all $U \in \mathcal{U}$. Clearly $\mathcal{U} \subset \bigcup \{\mathcal{G}_x: x \in \bigcup_{\alpha<\beta+1} Y_\alpha\}$ and $p \in X - \bigcup \mathcal{U}$, but $Y_{\beta+1} \subset Y \subset \bigcup_{x \in D} \mathrm{st}(x, \mathcal{V})$ and hence $Y_{\beta+1} - \bigcup \mathcal{U} = \varnothing$, a contradiction.

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