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SHORTER NOTES

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INTERPOLATION BY A FINITE BLASCHKE PRODUCT[1]

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In this note, we present constructive proofs of the following two interpolation results:

THEOREM. *If $z_1, \ldots, z_n$ are distinct real numbers and if $w_1, \ldots, w_n$ are arbitrary real numbers, then there exists a rational function f with real poles which maps the upper-half plane to itself and satisfies $f(z_i) = w_i$ for $i = 1, \ldots, n$.*

A finite Blaschke product is a complex function of the form

$$B(z) = \lambda \prod_{i=1}^{m} \left(\frac{z - a_i}{1 - \bar{a}_i z} \right),$$

where $|\lambda| = 1$, $|a_i| < 1$, $i = 1, \ldots, m$ and $m \geqslant 0$.

COROLLARY. *If $\alpha_1, \ldots, \alpha_n$ are distinct complex number of modulus one and if $\beta_1, \ldots, \beta_n$ have modulus one, then there exists a finite Blaschke product B such that $B(\alpha_i) = \beta_i$ for $i = 1, \ldots, n$.*

An algebraic proof of the existence of the function f in the above theorem is implicitly given in [3], and an independent proof of its corollary was given in [2]. A recent preprint of M. B. Abrahamse and S. D. Fisher [1] includes an existence proof of the function f in the above theorem.

PROOF OF THE THEOREM. Let

$$p_k(z) = -w_k - \sum_{\substack{i=1 \\ i \neq k}}^{n} \left(\frac{1}{z_i - z_k} \right) + \sum_{\substack{i=1 \\ i \neq k}}^{n} \left(\frac{1}{z_i - z} \right),$$

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[1]The proofs in this note appeared in the author's doctoral thesis written at the University of Georgia under the direction of Professor K. Clancey.

and $g_k(z) = -w_k^2/z$ for $1 \leqslant k \leqslant n$. Note that each p_k has real zeros and each z_i $(i \neq k)$ is a pole for p_k. Each p_k and g_k maps the upper half-plane to itself. Hence each $f_k = g_k \circ p_k$ maps the upper half-plane to itself and satisfies $f_k(z_i) = 0$ $(i \neq k)$, $f_k(z_k) = w_k$. Set $f = \sum_{k=1}^n f_k$; then f is the required rational function.

Proof of the corollary. Without loss of generality, we can assume that α_i and β_i are different from 1 for $i = 1, \ldots, n$. Let f be as in the theorem which maps $\mathcal{O}(\alpha_i)$ to $\mathcal{O}(\beta_i)$ $(i = 1, \ldots, n)$, where $\mathcal{O}(z) = i(1+z)/(1-z)$, $|z| < 1$. The function $B = \mathcal{O}^{-1} \circ f \circ \mathcal{O}$ is the required finite Blaschke product.

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