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REDUCING THE CODIMENSION OF KÄHLER IMMERSIONS

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ABSTRACT. The codimension of an immersion of a Kähler manifold may be reduced if there is a holomorphic vector field normal to the manifold.

There have been several recent results on reducing the codimension of an isometric immersion (Erbacher [2], Yau [3, p. 351]) and in particular of a minimal immersion (Colares and do Carmo [1]). In connection with these, the following result for complex geometry may be of interest.

THEOREM. *Let M be a complex submanifold in $\mathbf{C}^n$ with normal bundle N and let V be an open subset of M. If $N|_V$ admits r holomorphic sections, then M lies in some $\mathbf{C}^{n-r}$.*

Recall that for any complex manifold U there is a splitting of the complexified tangent bundle $\mathbf{C} \otimes T(U) = T^{1,0}U \otimes T^{0,1}U$. For typographical convenience let us denote $T^{1,0}U$ by TU for $U = M$, V or $\mathbf{C}^n$ and the restriction of $T^{1,0}\mathbf{C}^n$ to a bundle over M (or V) by $T'M$ (or $T'V$). In the theorem N is the normal bundle of M in $T\mathbf{C}^n$, $N = \{\xi \in T'M$ such that $\langle Z, \xi\rangle = 0$ for all $Z \in TM\}$. Here we use the standard Kähler metric on $T\mathbf{C}^n$. N is a complex bundle over M but in general it is not a holomorphic bundle. Indeed, in an appropriate sense, it is an antiholomorphic subbundle of $T'M$.

We may assume that V is a coordinate patch with coordinates $z_1, \ldots, z_m$. Let $Z_k = \partial/\partial z_k$. Let $I = (i_1, \ldots, i_k)$ be a multi-index with nonnegative integer components and let σ be a section of $T'V$. Using the usual connection on $T\mathbf{C}^n$ we derive a new section $Z^I \vdash \sigma$ by taking the $|I|$-fold covariant derivative of σ. Here $|I| = i_1 + \cdots + i_k$ and Z^I means first differentiate i_k times with respect to Z_k, etc.

Choose some point $q \in V$. Define $S_q = \{\xi \in T'V_q | \xi = Z^I \vdash \sigma$, I some multi-index and σ some holomorphic section of $TV\}$. Then $S = \bigcup S_q$, the union taken over all points $q \in V$, is a subset of $T'V$. We shall soon see it is a subbundle.

A section τ of $T'V$ is said to be parallel if its covariant derivative in each direction is zero.

LEMMA. *Let τ be a parallel section of $T'V$. If τ is orthogonal to S at some point $q \in V$, then τ is orthogonal to S at all points of V.*

PROOF. Because τ is parallel we have (1) for any local section σ of $T'V$, $Z^I\langle \sigma, \tau\rangle = \langle Z^I \vdash \sigma, \tau\rangle$. This also holds for $\bar{Z}$, so (2) $\langle \sigma, \tau\rangle$ is a holomorphic function whenever σ is a local holomorphic section of $T'V$. Now if σ is a

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holomorphic section in a neighborhood of q then, since τ is orthogonal to S at q, $\langle Z^I \vdash \sigma, \tau \rangle = 0$ at q and so by (1), $Z^I \langle \sigma, \tau \rangle = 0$ at q for all I. By (2), $\langle \sigma, \tau \rangle = 0$ in this neighborhood. Now analytic continuation may be used to show that $\langle Z^I \vdash \sigma, \tau \rangle = 0$ whenever σ is a local holomorphic section of TV.

COROLLARY *S and its orthogonal complement are holomorphic subbundles over V of $T\mathbf{C}^n$ and each is invariant under parallel translation.*

PROOF. Let $S^\perp = \bigcup \{\xi \in T\mathbf{C}^n_q | \langle s, \xi \rangle = 0 \text{ for all } s \in S_q\}$, the union taken over all $q \in V$. Any $\xi \in T\mathbf{C}^n_q$ has a parallel extension. Therefore the Lemma implies that $S^\perp$ has constant fibre dimension and is invariant under parallel translation. The same must hold for S. But parallel sections are holomorphic. Thus both S and $S^\perp$ are holomorphic subbundles.

Now we have $T'V = S \oplus S^\perp$ and this decomposition is invariant under parallel translation. It follows easily that there is a compatible orthogonal decomposition $\mathbf{C}^n = \mathbf{C}^{n-p} \times \mathbf{C}^p$ where $p = \dim S^\perp$. Pick a point $q \in V \subset \mathbf{C}^n$. So $q = (q_1, q_2)$ with $q_1 \in \mathbf{C}^{n-p}$ and $q_2 \in \mathbf{C}^p$. Since $TV \subset S \subset T\mathbf{C}^{n-p}$, it follows that $V \subset \mathbf{C}^{n-p} \times \{q_2\}$. Now if we are given, as in the Theorem, r holomorphic sections of $T'V$ which are orthogonal to TV then $\dim S^\perp \geqslant r$ and so V is contained in some $\mathbf{C}^{n-r}$ and the same must be true for M.

The following observation relates this Theorem to the results of Erbacher and Yau. Let ξ be a holomorphic section of $T'V$ and assume ξ is orthogonal to TV. We write $\xi = U - iJU$ where U is in the real tangent space of $\mathbf{C}^n$ and J gives the complex structure. Then V as a real submanifold of $\mathbf{R}^{2n}$ is totally geodesic in the directions U and JU.

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