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THE LENGTH SPECTRUM OF A RIEMANN SURFACE IS ALWAYS OF UNBOUNDED MULTIPLICITY

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ABSTRACT. I show that the length spectrum of a Riemann surface is always of unbounded multiplicity, and indicate connections with recent work of Guillemin and Kazhdan.

V. Guillemin and D. Kazhdan have recently established [4] that if S is a compact surface having a metric of negative curvature, then under rather general conditions, the spectrum of the Schrödinger operator $\Delta + q(x)$ on S determines the function $q(x)$. The major hypothesis of their theorem is that the length spectrum of S be simple, i.e., no two distinct unoriented closed geodesics on S have the same length. It is known (cf. [1]) that this is true for most metrics. On the other hand, it is a curious fact that it is never true in the Riemann surface case, i.e., when S has constant curvature -1. In this note, we will show this and somewhat more, namely that in the constant curvature case, the length spectrum is never even of bounded multiplicity. I am told that in the same vein, T. Jørgensen has observed, using a geometric argument, that there are always present lengths of multiplicity at least 2.

We now pass to a proof of the result.

It follows from a result of R. Horowitz [5, p. 648], that if A and B are two elements of PSL(2, R) which generate a free group G, then for any positive integer N, there exist words $W_1, \ldots, W_N$ in A and B such that:

(1) None of the $W_1, \ldots, W_N$ are conjugate in G.

(2) No two of the $W_1, \ldots, W_N$ are inverses of one another.

(3) $|\operatorname{tr} W_1| = |\operatorname{tr} W_2| = \cdots = |\operatorname{tr} W_N|$.

Now the conjugacy problem for fundamental groups of compact orientable surfaces was solved in 1912 by Dehn [2]. Cf. also [3]. It is an immediate consequence of Dehn's paper (p. 420) that if $a_1, b_1, a_2, b_2, \ldots, a_g, b_g$, $(a_1 b_1 a_1^{-1} b_1^{-1}) \cdots (a_g b_g a_g^{-1} b_g^{-1}) = 1$, is a standard presentation of the fundamental group P of a surface of genus $g > 1$, and if x and y are two elements in the free group F generated by $a_1, \ldots, a_g$, then x and y are conjugate in P if and only if they are conjugate in F. Thus, if in the Horowitz example we take $x = a_1, y = a_2$, we immediately find, using the correspondence between matrices and geodesics, that the multiplicity of the length spectrum of a Riemann surface is never bounded. The same technique applies to the noncompact case, since the fundamental group is then a free group.

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