PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 78, Number 4, April 1980

# SMOOTHNESS AND WEAK* SEQUENTIAL COMPACTNESS

JAMES HAGLER[1] AND FRANCIS SULLIVAN

ABSTRACT. If a Banach space $E$ has an equivalent smooth norm, then every bounded sequence in $E^*$ has a weak* converging subsequence. Generalizations of this result are obtained.

**1. Introduction.** This paper examines some connections between geometric conditions on a Banach space $E$ and the following condition on $E$:

($\omega$) Every bounded sequence in $E^*$ has a weak* converging subsequence.

Our main result is:

THEOREM 1. *Suppose $F$ is a closed subspace of the Banach space $E$, $F$ has ($\omega$) while $E$ fails ($\omega$). Then $E/F$ has no equivalent smooth norm. In particular, a smooth Banach space has ($\omega$).*

Recall that the norm $\|\cdot\|$ on $E$ is smooth if for every $e \in E$, $e \neq 0$, there is a unique $f \in E^*$ with $\|f\| = 1$ and $f(e) = \|e\|$. Of course, $E$ has ($\omega$) if and only if the unit ball of $E^*$ is weak* sequentially compact.

The first indication that fairly "weak" geometric conditions on $E$ might imply ($\omega$) comes from the following result of Stegall **[15]**: A weak Asplund space has ($\omega$). (See §3 for a definition of weak Asplund space.) Previously, Asplund **[1]** had shown that if $E^*$ is strictly convex then $E$ is weak Asplund, and consequently, by Stegall's result, $E$ has ($\omega$). An elementary duality computation shows that if $E^*$ is strictly convex then $E$ is smooth, but counterexamples to the converse have been given by Klee **[8]** and Troyanski **[16]**. It is not known if every smooth space is weak Asplund. However, Theorem 1 does provide a partial answer to Stegall's question about the general relation between differentiability conditions and property ($\omega$) on a Banach space $E$. Also, using some ideas from **[5]** and **[6]**, J. Bourgain has constructed a closed subspace $X$ of $l^\infty$ with $c_0 \subset X$ such that $X$ fails ($\omega$) while $Y = X/c_0$ has ($\omega$). This shows that property ($\omega$) is *not* a three-space property. Moreover, this combines with Theorem 1 to show that $Y$ is a Banach space having ($\omega$) while (1) $Y$ has no equivalent smooth norm, and (2) $Y$ is not a weak Asplund space. (Assertion (1) follows directly from Theorem 1 while the proof of (2) uses Lemma 3 below.) We include this example with Professor Bourgain's kind permission.

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Presented to the Society January 26, 1979; received by the editors November 14, 1978 and, in revised form, May 1, 1979.

*AMS (MOS) subject classifications* (1970). Primary 46B99.

[1]Research of the first named author was partially supported by the National Science Foundation.

The proof of Theorem 1 is based on two ideas. The first is a completeness argument of Leach and Whitfield [10] (cf. also John and Zizler [7]) which is similar to the proof of the Bishop-Phelps Theorem [2]. (In fact, if the continuum hypothesis is assumed, an easy proof that a smooth space has $(\omega)$ follows directly from the Bishop-Phelps Theorem.) The completeness argument shows that if some equivalent norm for $E$ is strongly rough (see §2) then no equivalent norm for $E$ can be smooth.

The second idea is based on a diagonalization technique used by Stegall [15] to prove that a weak Asplund space has $(\omega)$. Larman and Phelps [9] gave a refinement of Stegall's argument which shows that $E$ has $(\omega)$ if each nonempty weak* compact convex subset of $E^*$ has a "weak* $G_\delta$ extreme point". A modification of the Larman-Phelps argument is used in the proof of Theorem 1.

§2 contains definitions and the proof of Theorem 1. In §3, we sketch the easy proof using the continuum hypothesis that a smooth space has $(\omega)$. In addition, we discuss briefly our results in relation to Gâteaux differentiability of convex functions on $E$, and present the construction due to J. Bourgain mentioned earlier.

We wish to thank the referee for showing us a proof of Lemma 3 which illuminates clearly its connection with the Bishop-Phelps theorem, as well as for pointing out that our original proof of Theorem 1 could be recast in terms of weak* $G_\delta$ extreme points.

**2. Bounded sequences in $E^*$.** Recall that smoothness of the norm in $E$ is equivalent to saying that, for each $\|x\| = 1$ and all $y \in E$,

$$\lim_{t\to 0+} \frac{\|x + ty\| + \|x - ty\| - 2}{t} = 0.$$

Following Leach and Whitfield we say that the norm in $E$ is *strongly rough* if it uniformly avoids being smooth; e.g. there is an $\varepsilon > 0$ such that for each $\|x\| = 1$ there is a $\|y\| \leqslant 1$ with

$$\limsup_{t\to 0+} \frac{\|x + ty\| + \|x - ty\| - 2}{t} > \varepsilon.$$

The following characterization is implicit in the work of Leach and Whitfield [10] and John and Zizler [7] and will be used in the sequel.

LEMMA 2. *The norm in $E$ is strongly rough if, and only if, there is an $\varepsilon > 0$ such that for each $z \in E$ there is a $\|v\| = 1$ satisfying*

$$\|z + tv\| \geqslant \|z\| + |t|\varepsilon \quad \textit{for all real } t. \qquad (*)$$

PROOF. Suppose that the norm in $E$ is strongly rough and let $\bar{\varepsilon}$ be the "degree" of roughness occurring in the definition. Let $\varepsilon = \bar{\varepsilon}/4$ and suppose $z \in E$ is arbitrary. If $z = 0$ then any $\|v\| = 1$ will do for $(*)$. Otherwise let $x = z/\|z\|$. Using a simple duality argument and the definition of strongly rough we get that there are $f, g \in E^*$, $\|f\| = 1 = \|g\|$, such that $f(x) = 1 = g(x)$ while for some $\|y\| = 1$, $(f - g)(y) > \bar{\varepsilon}$.

Let $w = y - (((f + g)(y))/2)x$ and notice that $\|w\| \leqslant 2$. If now $v = w/\|w\|$ and $t > 0$ then

$$\|z + tv\| \geqslant f(z) + tf(v) = \|z\| + t\frac{(f-g)(w)}{2\|w\|}$$

$$> \|z\| + t\frac{\bar{\varepsilon}}{2\|w\|} \geqslant \|z\| + |t|\varepsilon.$$

The case for $t < 0$ is handled similarly.

The converse is clear. Q.E.D.

It is obvious that, for $\Gamma$ uncountable, the norm in $l^1(\Gamma)$ is strongly rough and it is not hard to show that $l^\infty/c_0$ has an equivalent strongly rough norm. It is known that neither of these spaces can be renormed for smoothness. The following lemma shows that this is a general phenomenon. The original proof was due to Leach and Whitfield and the enlightening simplification given here has been supplied by the referee.

LEMMA 3. *If $E$ has an equivalent smooth norm, then $E$ has no equivalent strongly rough norm.*

PROOF. We show first that if $\|\cdot\|$ is an equivalent strongly rough norm for $E$ and $\rho$: $E \to \mathbf{R}$ is any Gâteaux differentiable function with $\rho(0) = 0$, then the nonempty set $S \equiv \{x | \rho(x) \leqslant \|x\|\}$ is unbounded. Assume that $S$ is bounded and define a partial order on $E$ by $x > y$ if $\varepsilon\|x - y\| \leqslant \|x\| - \|y\|$. Here $\varepsilon > 0$ is from (*) in Lemma 2. As in the proof of the Bishop-Phelps theorem and its generalizations **[11]**, we can use completeness to obtain a maximal $x_0 \in S$. Since $\|\cdot\|$ is strongly rough there is a $\|y\| = 1$ such that for all $t$, $\|x_0 + ty\| \geqslant \|x_0\| + |t|\varepsilon$. In other words, for all $t$, $x_0 + ty > x_0$. Since $x_0$ is maximal $x_0 + ty \notin S$ and so if $t \neq 0$, $\rho(x_0 + ty) > \|x_0 + ty\|$. Combining this with the fact that $\rho(x_0) \leqslant \|x_0\|$ we have that, for all $t$,

$$\rho(x_0 + ty) - \rho(x_0) \geqslant \|x_0 + ty\| - \|x_0\| \geqslant \varepsilon|t|.$$

This contradicts the linearity of $\rho'(x_0)$.

If $E$ has an equivalent smooth norm $|\cdot|$, then there is a constant $M$ with $\|x\| \leqslant M|x|$ for all $x$. On the other hand, if we define $\rho(x) = |x|^2$, then $\rho$ is Gâteaux differentiable and $\rho(0) = 0$. Thus, $\{x| \, |x|^2 \leqslant M|x|\} \supset \{x| \, |x|^2 \leqslant \|x\|\}$ is unbounded (in the norm $\|\cdot\|$), which contradicts the equivalence of $\|\cdot\|$ and $|\cdot|$. Q.E.D.

Following Larman and Phelps **[9]** we say that $E^*$ has $\omega^*$ $G_\delta$ extreme points if every nonempty weak* compact convex $K \subset E^*$ has an extreme point which is a $G_\delta$ point in the relativized weak* topology on $K$. It is easy to show that if $E$ has an equivalent smooth norm, then $E^*$ has weak* $G_\delta$ extreme points. Indeed, let $K$ be a nonempty weak* compact convex subset of $E^*$. Then $K$ has weak* faces of small norm diameter, i.e. for each $\varepsilon > 0$, there is an $\|x\| = 1$ such that $\operatorname{diam}\{f \in K | f(x) = \sup\{g(x): g \in K\}\} < \varepsilon$. Once this is shown, then we may select a sequence $K \supset K_1 \supset K_2 \supset \cdots$ where each $K_i$ is weak* compact and convex, $\operatorname{diam}(K_i) < 1/i$ and $K_i$ is a weak* face of $K_{i-1}$. It is immediate that $\bigcap_i K_i$ is a weak* $G_\delta$ extreme point of $K$.

To establish the assertion about small faces, suppose that $K$ does not have weak* faces of diameter less than $\varepsilon$. Then if $B^*$ is the unit ball of $E^*$, the set $A = B^* + K - K$ is weak* compact, convex, balanced and has nonempty norm interior. It is not hard to check that the Minkowski functional for $A$ defines an equivalent norm on $E^*$ dual to a strongly rough norm on $E$. But Lemma 3 shows that this cannot happen, since $E$ has an equivalent smooth norm.

REMARK. Much of the proof of Theorem 1 is a repetition of Proposition 12 of **[9]**, where it is shown that $E$ has $(\omega)$ if $E^*$ has weak* $G_\delta$ extreme points. We include this for completeness.

PROOF OF THEOREM 1. Suppose $(g_n)$ is a bounded sequence in $E^*$. Since $F$ has $(\omega)$, there is a subsequence $(h_n)$ of $(g_n)$ and an $h \in F^*$ such that $h_n|_F \to h$ weak* in $F^*$. Let $f$ be a Hahn-Banach extension of $h$ to $E$. If we define $f_n = h_n - f$, then $(f_n)$ is a bounded sequence in $E^*$ and *every* weak* accumulation point of $\{f_n : n \in N\}$ is in $F^\perp$.

Following the notation of **[4]**, we symbolize $(f_n)$ by its ordered set of indices, $N$. Subsequences can then be indicated by $M$, $L \subset N$, etc. Here, for instance, if $M = \{n_1, n_2, \dots\}$, then $M$ "stands for" the subsequence $(f_{n_i})$ of $(f_n)$. The set of weak* accumulation points of the subsequence $M$ will be denoted by $M'$.

Now, let $K = \overline{\text{co}}^*(N')$. Since $E/F$ has an equivalent smooth norm, the remarks above and the weak* continuous isometry between $(E/F)^*$ and $F^\perp$ yield a point $g \in N'$ which is a weak* $G_\delta$ extreme point of $K$. Thus there exist weak* open *in* $E^*$ sets $U_1 \supset U_2 \supset \cdots$ such that $\overline{U_i}^* \subset U_{i-1}$ and $\bigcap_i (U_i \cap K) = \{g\}$. Pick subsequences $N \supset N_1 \supset N_2 \supset \cdots$ such that $N_i \subset U_i$ for each $i$. Let $(f_{n_i})$ be a subsequence of $(f_n)$ satisfying $f_{n_i} \in N_i$ for each $i$. Then

$$\{f_{n_i} : i \in N\}' \subset \bigcap_{i=1}^{\infty} N_i' \subset \bigcap_{i=1}^{\infty} (\overline{U_i}^* \cap K) = \{g\}.$$

So $f_{n_i}$ converges weak* to $g$. Thus, the subsequence $(h_{n_i})$ of $(g_n)$ converges weak* to $f + g$. This completes the proof of Theorem 1.

**3. Remarks.** Let us first sketch the simple proof (assuming the continuum hypothesis) that a smooth space has $(\omega)$. Observe that if $E$ is smooth, then so is every subspace, and if $E$ fails $(\omega)$, then there is a subspace $E_0$ of $E$ of dimension (= density character) $\mathfrak{c}$ with $E_0$ failing $(\omega)$. To see this, let $(f_n)$ be a bounded sequence in $E^*$ with no weak* converging subsequence, and for each subsequence $M \subset N$, pick $x_M \in E$ such that $\lim_{n \in M} f_n(x_M)$ does not exist. Then the subspace $E_0$ of $E$ generated by the set $\{x_M : M \text{ is a subsequence of } N\}$ has the desired properties.

So we may assume that $E$ is smooth, fails $(\omega)$, and has dimension $\mathfrak{c}$. By smoothness and the Bishop-Phelps Theorem, the norming map $n: E \to E^*$ defined by $n(e)(e) = \|e\|^2$, $\|n(e)\| = \|e\|$ is one-to-one and has norm dense range. Thus, $\text{card}(E^*) \leqslant (\text{card}(n(E)))^{\aleph_0} = (\text{card}(E))^{\aleph_0} = \mathfrak{c}^{\aleph_0} = \mathfrak{c}$, since a Banach space of dimension $\mathfrak{c}$ has cardinality $\mathfrak{c}$.

On the other hand, by the Čech-Pospisil Theorem [3], any compact space which is not sequentially compact has cardinality $\geqslant 2^{\aleph_1}$. Assuming the continuum hypothesis, and applying the Čech-Pospisil Theorem to $B^*$, we have that card$(E^*) \geqslant 2^{\mathfrak{c}}$. This is a contradiction.

REMARK. We do not know if, even with the continuum hypothesis, our results about quotients can be deduced directly from the Bishop-Phelps Theorem.

We now briefly connect our results with the notion of Gâteaux differentiability. Recall that a Banach space $E$ is called a weak Asplund space if every continuous convex function on $E$ is Gâteaux differentiable at least at a dense $G_\delta$ subset of its domain. As mentioned above, Stegall proved that a weak Asplund space has $(\omega)$. Extending this, Phelps [12] proved that a Banach space has $(\omega)$ if every continuous guage on $E$ is Gâteaux differentiable at at least one point. Our main result can be restated as follows: $E$ has $(\omega)$ if every equivalent norm $\|\cdot\|$ on $E$ is $\varepsilon$-differentiable at some point, i.e. if for every $\varepsilon > 0$, there exists an $\|x\| = 1$ such that for all $\|y\| \leqslant 1$,

$$\limsup_{t\to 0+} \frac{\|x + ty\| + \|x - ty\| - 2}{t} < \varepsilon.$$

At present, we do not know if this implies that every equivalent norm on $E$ has a point of Gâteaux differentiability. Of course, the existence of a strongly rough norm is precisely the condition that this norm is *uniformly* non-Gâteaux differentiable at every point.

Finally, we outline the example of Bourgain mentioned in the Introduction. (This is essentially the same example given by Haydon in [6] for another purpose.) Let $N$ denote the set of natural numbers. The following is proved in [4].

LEMMA. *There is a well ordered set* $(I, <)$ *and a collection* $(M_\alpha)_{\alpha\in I}$ *of infinite subsets of* $N$ *such that* (1) *if* $\alpha < \beta$ *then either* $M_\beta \subset_a M_\alpha$ *or* $M_\beta \cap M_\alpha =_a \varnothing$; *and* (2) *if* $M \subset N$ *is infinite, then there is an* $\alpha \in I$ *such that both* $M \cap M_\alpha$ *and* $M \setminus M_\alpha$ *are infinite.*

(We say $K \subset_a L$ if $K \setminus L$ is finite and $K \cap L =_a \varnothing$ if $K \cap L$ is finite.)

For each $\alpha$, let $\varphi_\alpha$ be the indicator function of $M_\alpha$, i.e., $\varphi_\alpha(n) = 1$ if $n \in M_\alpha$ and 0 otherwise. The Banach space $X$ is the closed subspace of $l^\infty$ spanned by $\{\varphi_\alpha : \alpha \in I\} \cup c_0$. Let $f \in X^*$ be defined by $f_n(\xi) = \xi_n$ for $\xi = (\xi_k) \in X$. Since $c_0 \subset X$, $\|f_n\| = 1$ for each $n$, and a simple calculation (as in [4]) using property (2) of the lemma shows that $(f_n)$ has no weak* converging subsequence. Thus, $X$ fails $(\omega)$.

We now show that $X/c_0$ has $(\omega)$. Suppose $(g_n)$ is a norm-one sequence in $(X/c_0)^*$ with no weak* converging subsequence. Let $\hat{\varphi}_\alpha = \varphi_\alpha + c_0 \in X/c_0$. Then since $T = \{\hat{\varphi}_\alpha : \alpha \in I\}$ generates $X/c_0$, it follows that $(g_n|_T)$ is a uniformly bounded sequence of functions on $T$ with no pointwise converging subsequence. Hence, by the results of [13] (cf. also [14]) there exists a subsequence $(f_n)$ of $(g_n)$ and real numbers $r$ and $\delta > 0$ such that $(A_n, B_n)$ form an independent family of pairs of sets, where $A_n = \{t \in T : f_n(t) > r + \delta\}$ and $B_n = \{t \in T : f_n(t) < r\}$. For fixed $n$, pick

$$t_1 \in \left( \bigcap_{i=1}^{2^{n-1}} A_i \right) \cap \left( \bigcap_{i=2^{n-1}+1}^{2^n} B_i \right),$$

$$t_2 \in \left( \bigcap_{i=1}^{2^{n-2}} A_i \right) \cap \left[ \bigcap_{i=2^{n-2}+1}^{2^{n-1}} B_i \right] \cap \left[ \bigcap_{i=2^{n-1}+1}^{3\cdot 2^{n-2}} A_i \right] \cap \left( \bigcap_{i=3\cdot 2^{n-2}+1}^{2^n} B_i \right)$$

$$\vdots$$

$$t_n \in A_1 \cap B_2 \cap A_3 \cap \cdots \cap A_{2^n-1} \cap B_{2^n}.$$

Now, each $t_i = \hat{\varphi}_{\alpha_i}$ for some $\alpha_i \in I$. A calculation as in **[13]** shows that

$$\left\| \sum_{i=1}^{n} a_i \hat{\varphi}_{\alpha_i} \right\| > \delta \sum_{i=1}^{n} \left| \frac{a_i}{2} \right|$$

for all scalars $a_1, \ldots, a_n$, i.e. that the set $\{\hat{\varphi}_{\alpha_i}: i = 1, \ldots, n\}$ is $\delta/2$ equivalent to the usual basis of $l_n^1$.

Notice, however, that if $M_{\beta_1}, \ldots, M_{\beta_k}$ are pairwise almost disjoint then

$$\left\| \sum_{i=1}^{k} a_i \hat{\varphi}_{\beta_i} \right\| = \max_i |a_i|,$$

while if $M_{\beta_1} \supset_a M_{\beta_2} \supset_a \cdots \supset_a M_{\beta_k}$, then

$$\left\| \sum_{i=1}^{k} a_i \hat{\varphi}_{\beta_i} \right\| = \max_i \left| \sum_{j=1}^{i} a_j \right|.$$

Hence for $k$ sufficiently large (depending on $\delta$) neither of these sets can be $\delta/2$-equivalent to the usual basis of $l_k^1$. On the other hand, for $n$ sufficiently large (say $n > k^k$) one of these two situations must occur for *any* choice of $n$ cosets $\hat{\varphi}_{\alpha_1}, \ldots, \hat{\varphi}_{\alpha_n}$. Thus, no $n$ elements subset of $\{\hat{\varphi}_\alpha: \alpha \in I\}$ can be $\delta/2$ equivalent to the usual basis of $l_n^1$. This contradiction shows that $X/c_0$ has property $(\omega)$.

REMARK. Larman and Phelps **[9]** prove that $E^*$ having weak* $G_\delta$ extreme points *is* a three space property. Since $X$ fails $(\omega)$ we have that $(X/c_0)$ has $(\omega)$ but $(X/c_0)^*$ does not have weak* $G_\delta$ extreme points.

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DEPARTMENT OF MATHEMATICS, CATHOLIC UNIVERSITY OF AMERICA, WASHINGTON, D. C. 20064