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MONOTONE ITERATION AND GREEN'S FUNCTIONS FOR BOUNDARY VALUE PROBLEMS

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ABSTRACT. An iteration scheme is given for approximating solutions of boundary problems of the form $Ly = f(x, y)$, $Ty(x) = r$, where L is an nth order linear differential operator, f is continuous and T is a continuous linear operator from $C^{n-1}(I)$ into $\mathbf{R}^n$. The scheme is based on the condition that the Green's function $G(x, s)$ for the associated linear problem $Ly = 0$, $Ty = 0$ exists and has sign independent of s.

1. Introduction. Let $n \geqslant 1$, let $I = [a, b]$ be a real interval, let $a = x_1 < x_2 < \cdots < x_k = b$, let $p_1(x), p_2(x), \ldots, p_n(x)$ be continuous on I, and define the linear differential operator L by

$$Ly = y^{(n)} + p_1(x)y^{(n-1)} + \cdots + p_n(x)y. \tag{1.1}$$

A. Ju. Levin (see [**1**]) has obtained the following result.

THEOREM L. *Let L and I be as above, and suppose that L is disconjugate on I. Then the Green's function $G(t, s)$ for the k-point boundary value problem (BVP)*

$$Ly = 0, \tag{1.2}$$

$$y^{(i)}(x_j) = 0, \qquad i = 0, \ldots, n_j - 1, j = 1, \ldots, k, \tag{1.3}$$

where $\sum n_j = n$, satisfies the inequality

$$G(x, s)(x - x_1)^{n_1}(x - x_2)^{n_2} \cdot \cdots \cdot (x - x_k)^{n_k} \geqslant 0, \qquad x_1 < s < x_k. \tag{1.4}$$

For our purposes, the importance of Levin's theorem is that in this instance the following condition holds.

CONDITION S. There exists a Green's function $G(x, s)$ for the problem $Ly = 0$, $Ty = 0$, and the sign of $G(x, s)$ is independent of s.

We present here a bilateral iteration scheme, based on Condition S, which will provide approximants to solutions of BVP's with linear boundary conditions. A. C. Peterson has found [**4**] that complete disconjugacy is not necessary for this condition to hold; he has recently shown [**5**] that it will also hold for certain q-focal problems, and we shall discuss these in the last section.

2. Linear boundary value problems. Let I be a real interval, let f: $I \times \mathbf{R} \to \mathbf{R}$ be continuous, and L be given by (1.1). Consider the BVP

$$Ly = f(x, y) \tag{2.1}$$

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with boundary conditions

$$Ty(x) = r, \tag{2.2}$$

where T: $C^{n-1}(I) \to \mathbf{R}^n$ is a continuous linear operator, r is a given constant vector. Assume that Condition S holds for the associated homogeneous problem $Ly = 0$, $Ty(x) = 0$. Hence there exist subsets of I, I_1 and I_2 such that

(i) $I = I_1 \cup I_2$ (possibly $I = I_1$ or $I = I_2$) and

(ii) $G(x, s)$ has sign given by

$$\begin{aligned} G(x, s) &\leqslant 0 \quad \text{for } a < s < b, x \in I_1, \\ G(x, s) &\geqslant 0 \quad \text{for } a < s < b, x \in I_2. \end{aligned} \tag{2.3}$$

Assume that there exists a constant M such that, for all (x, y_1), (x, y_2) in $I \times \mathbf{R}$,

$$|f(x, y_1) - f(x, y_2)| \leqslant M|y_1 - y_2|. \tag{2.4}$$

Further, suppose that there exist functions $v_1(x)$, $w_1(x)$ with piecewise continuous nth derivatives on I, such that

$$\begin{aligned} Tv_1(x) = Tw_1(x) = r, &\quad \text{and such that, for } x \in I, \\ Lv_1 - f(x, v_1) + A_1(x) &\equiv \beta_1(x) \leqslant 0, \\ Lw_1 - f(x, w_1) - A_1(x) &\equiv \gamma_1(x) \geqslant 0, \end{aligned} \tag{2.5}$$

where

$$A_1(x) = M|v_1(x) - w_1(x)|. \tag{2.6}$$

Let $l_r(x)$ be the solution of the problem $Ly = 0$, $Ty(x) = r$; existence of $l_r(x)$ follows from linearity and uniqueness [2]. Construct the sequences $\{v_m(x)\}$ and $\{w_m(x)\}$ as follows:

$$\begin{aligned} v_{m+1}(x) &= l_r(x) + \int_I G(x, s)[f(s, v_m(s)) - A_m(s)]\, ds, \\ w_{m+1}(x) &= l_r(x) + \int_I G(x, s)[f(s, w_m(s)) + A_m(s)]\, ds, \end{aligned} \tag{2.7}$$

where

$$A_m(x) = M|v_m(x) - w_m(x)|, \qquad m \geqslant 1. \tag{2.8}$$

THEOREM 1. *Let L and f be as above; let* (2.4) *and Condition* S *hold. Suppose that there exist functions* $v_1(x)$ *and* $w_1(x)$ *satisfying* (2.5), *and define the sequences* $\{v_m(x)\}$ *and* $\{w_m(x)\}$ *by* (2.7). *Then there exists a solution* $y(x)$ *of the BVP* (2.1)–(2.2) *such that, for all* $m \geqslant 1$,

$$\begin{aligned} v_m(x) \geqslant v_{m+1}(x) \geqslant y(x) \geqslant w_{m+1}(x) \geqslant w_m(x), &\qquad x \in I_1, \\ v_m(x) \leqslant v_{m+1}(x) \leqslant y(x) \leqslant w_{m+1}(x) \leqslant w_m(x), &\qquad x \in I_2. \end{aligned} \tag{2.9}$$

PROOF. Set $u_m(x) = v_m(x) - w_m(x)$, $m \geqslant 1$. Note that $u_1(x) \geqslant 0$ for $x \in I_1$, $u_1(x) \leqslant 0$ for $x \in I_2$, since $Lu_1 = f(x, v_1) - f(x, w_1) - 2A_1(x) + \beta_1 - \gamma_1 \leqslant 0$; hence $u_1(x) = \int_I G(x, s) Lu_1(s)\, ds$ has sign opposite to that of G. The rules for constructing the sequences $\{v_m(x)\}$ and $\{w_m(x)\}$ imply, similarly, that

$$u_{m+1}(x) = \int_I G(x, s)\left[f(s, v_m(s)) - f(s, w_m(s)) - 2M|v_m(s) - w_m(s)|\right] ds$$

and, from (2.4), we have that, for each $m \geq 1$,

$$u_{m+1}(x) \geq -\int_I G(x, s) M|u_m(s)|\, ds \geq 0, \qquad x \in I_1, \tag{2.10a}$$

and

$$u_{m+1}(x) \leq -\int_I G(x, s) M|u_m(s)|\, ds \leq 0, \qquad x \in I_2. \tag{2.10b}$$

Hence, for each $m \geq 1$,

$$v_m \geq w_m, \quad x \in I_1; \qquad v_m \leq w_m, \quad x \in I_2. \tag{2.11}$$

To show the monotonicity of the sequences $\{v_m(x)\}$, $\{w_m(x)\}$ on I_1 and I_2, note that $\beta_1 = Lv_1 - Lv_2$, $\gamma_1 = Lw_1 - Lw_2$, and set $\beta_m = Lv_m - Lv_{m+1}$, $\gamma_m = Lw_m - Lw_{m+1}$. Using (2.11), we can write

$$\beta_{m+1}(x) = \begin{cases} f(x, v_m) - f(x, v_{m+1}) - M(v_m - v_{m+1}) + M(w_m - w_{m+1}), & x \in I_1, \\ f(x, v_m) - f(x, v_{m+1}) + M(v_m - v_{m+1}) - M(w_m - w_{m+1}), & x \in I_2, \end{cases}$$

$$\gamma_{m+1}(x) = \begin{cases} f(x, w_m) - f(x, w_{m+1}) - M(w_m - w_{m+1}) + M(v_m - v_{m+1}), & x \in I_1, \\ f(x, w_m) - f(x, w_{m+1}) + M(w_m - w_{m+1}) - M(v_m - v_{m+1}), & x \in I_2. \end{cases}$$

Setting $\delta_m = v_m - v_{m+1}$, $\rho_m = w_m - w_{m+1}$, we obtain, using (2.4), the inequalities

$$\begin{aligned} \beta_{m+1}(x) &\leq M|\delta_m| - M\delta_m + M\rho_m, && x \in I_1, \\ \beta_{m+1}(x) &\leq M|\delta_m| + M\delta_m - M\rho_m, && x \in I_2; \\ \gamma_{m+1}(x) &\geq -M|\rho_m| - M\rho_m + M\delta_m, && x \in I_1, \\ \gamma_{m+1}(x) &\geq -M|\rho_m| + M\rho_m - M\delta_m, && x \in I_2. \end{aligned} \tag{2.12}$$

Since $L\rho_1 = \gamma_1$ and $\gamma_1 \geq 0$, with $T\rho_1 = 0$, it follows that $\rho_1 \leq 0$ on I_1, $\rho_1 \geq 0$ on I_2. Similarly, $\delta_1 \geq 0$ on I_1, $\delta_1 \leq 0$ on I_2. By (2.12), $\gamma_2 \geq 0$, $\beta_2 \leq 0$ on I, and, by induction, for each $m \geq 1$, $\rho_m \leq 0$ on I_1, $\rho_m \geq 0$ on I_2; $\delta_m \geq 0$ on I_1, $\delta_m \leq 0$ on I_2, and $\gamma_m \geq 0$, $\beta_m \leq 0$ on all of I. Hence $v_{m+1} \leq v_m$ and $w_{m+1} \geq w_m$ on I_1, $v_{m+1} \geq v_m$ and $w_{m+1} \leq w_m$ on I_2, and we have obtained the inequalities involving the v's and w's in (2.9). It remains to show that a solution $y(x)$ lies between the v's and w's. To prove this, note first that, on I_1 and on I_2, the sequences $\{v_m(x)\}$ and $\{w_m(x)\}$ are monotonic, bounded and equicontinuous. By Ascoli's theorem, they have uniform limits $v(x)$ and $w(x)$ with $v(x) \geq w(x)$ for $x \in I_1$, $v(x) \leq w(x)$ for $x \in I_2$. It follows from (2.7) that

$$Lv(x) = f(x, v) - A(x), \qquad Lw(x) = f(x, w) + A(x), \tag{2.13}$$

where $A(x) = M|v(x) - w(x)|$. Note that $A(x)$ is continuous and nonnegative and that $Tv(x) = Tw(x) = r$.

For each function $y(x) \in C(I)$, set

$$\bar{y}(x) = \begin{cases} \left.\begin{array}{ll} v(x) & \text{if } y(x) > v(x) \\ y(x) & \text{if } v(x) \geqslant y(x) \geqslant w(x) \\ w(x) & \text{if } y(x) < w(x) \end{array}\right\}, & x \in I_1, \\ \left.\begin{array}{ll} v(x) & \text{if } y(x) < v(x) \\ y(x) & \text{if } v(x) \leqslant y(x) \leqslant w(x) \\ w(x) & \text{if } y(x) > w(x) \end{array}\right\}, & x \in I_2, \end{cases}$$

and define $\hat{F}(x, y(x)) = f(x, \bar{y}(x))$. The function $\hat{F}$ is continuous and bounded on $I \times \mathbf{R}$. It follows from the Schauder fixed point theorem that the problem

$$Ly = \hat{F}(x, y), \qquad Ty(x) = r$$

has a solution $y(x)$. We now show that this solution satisfies

$$\begin{aligned} v(x) \geqslant y(x) \geqslant w(x), &\qquad x \in I_1, \\ v(x) \leqslant y(x) \leqslant w(x), &\qquad x \in I_2, \end{aligned}$$

and hence that $y(x)$ is a solution of (2.1)–(2.2). Let D be the compact domain bounded by $v(x)$, $w(x)$ and the lines $x = a$ and $x = b$. Set $z(x) = w(x) - y(x)$. Then

$$\begin{aligned} Lz(x) &= Lw(x) - Ly(x) \\ &= f(x, w(x)) + M|v(x) - w(x)| - f(x, \bar{y}(x)) \geqslant 0 \end{aligned}$$

since (2.4) holds. Furthermore, since $Tz(x) = 0$, we have $z(x) \leqslant 0$, $x \in I_1$, $z(x) \geqslant 0$, $x \in I_2$. Similarly, setting $\hat{z}(x) = v(x) - y(x)$, we obtain $\hat{z}(x) \geqslant 0$, $x \in I_1$, $\hat{z}(x) \leqslant 0$, $x \in I_2$. Hence $(x, y(x))$ lies in D for all $x \in I$, and the proof is complete.

REMARKS. (i) It is necessary only that the bound (2.4) hold for all $(x, y) \in D^{(1)}$, where $D^{(1)}$ is the compact domain bounded by the curves $v_1(x)$, $w_1(x)$ and the lines $x = a$ and $x = b$.

(ii) Set $G = \max_{x \in I} |\int_I G(x, s)\, ds|$. Then if $2MG < 1$, and if $|f(x, y)| < B$, for some constant B for all $(x, y) \in I \times \mathbf{R}$, the functions v_1 and w_1 can be chosen as

$$\begin{aligned} v_1(x) &= l_r(x) - \frac{B}{1 - 2MG} \int_I G(x, s)\, ds, \\ w_1(x) &= l_r(x) + \frac{B}{1 - 2MG} \int_I G(x, s)\, ds. \end{aligned}$$

(iii) In case f has certain monotonicity properties, the Lipschitz continuity is not needed, and the iteration can be simplified by taking $A_i(x) \equiv 0$ for all $i \geqslant 1$. Furthermore, the functions v_1 and w_1 can be readily obtained from $G(x, s)$ and $l_r(x)$ as before, but now without the requirement that $2MG < 1$. Inspection of the proof of Theorem 1 leads to the following result.

THEOREM 2. *Let L and T be as in Theorem* 1. *Let $f(x, y)$ be continuous on $I \times \mathbf{R}$, and be monotone decreasing in y for each $x \in I$ and monotone increasing in y for each $x \in I_2$. Then if there exist functions $v_1(x)$ and $w_1(x)$ satisfying*

$$v_1 \geqslant w_1, \quad x \in I_1; \qquad v_1 \leqslant w_1, \quad x \in I_2,$$

$$Tv_1(x) = Tw_1(x) = r, \quad \text{and such that, for } x \in I,$$

$$Lv_1 - f(x, v_1) \equiv \beta_1(x) \leqslant 0,$$

$$Lw_1 - f(x, w_1) \equiv \gamma_1(x) \geqslant 0,$$

and if the sequences $\{v_m(x)\}$, $\{w_m(x)\}$ are defined by

$$v_{m+1}(x) = l_r(x) + \int_I G(x, s) f(s, v_m(s))\, ds,$$

$$w_{m+1}(x) = l_r(x) + \int_I G(x, s) f(s, w_m(s))\, ds, \tag{2.14}$$

these sequences will converge to solutions $v(x)$ and $w(x)$ of the BVP (2.1)–(2.2), *and*

$$v_m(x) \geqslant v_{m+1}(x) \geqslant v(x) \geqslant w(x) \geqslant w_{m+1}(x) \geqslant w_m(x), \qquad x \in I_1,$$

$$v_m(x) \leqslant v_{m+1}(x) \leqslant v(x) \leqslant w(x) \leqslant w_{m+1}(x) \leqslant w_m(x), \qquad x \in I_2.$$

Further, any solution $y(x)$ of the BVP (2.1)–(2.2) *which lies between v_1 and w_1 will also lie between v and w. In case $|f(x, y)| \leqslant B$ for some constant B for all $(x, y) \in I \times \mathbf{R}$, the functions v_1 and w_1 can be chosen as*

$$v_1(x) = l_r(x) - B \int_I G(x, s)\, ds,$$

$$w_1(x) = l_r(x) + B \int_I G(x, s)\, ds.$$

3. Applications. We consider two applications.

(i) Let L in (2.1) be disconjugate on I, and suppose that the boundary conditions (2.2) are the conjugate boundary conditions

$$y^{(i)}(x_j) = c_{ij}, \qquad 0 \leqslant i \leqslant n_{j-1}, j = 1, \ldots, k,$$

where $\sum n_j = n$, $a = x_1 < x_2 < \cdots < x_n = b$, and c_{ij} are constants. Levin's inequality (1.3) shows that I_1 will be the union of all subintervals $[x_j, x_{j+1}]$ of I such that $n_{j+1} + \cdots + n_k$ is odd and I_2 will be the union of all such subintervals such that the same sum is even.

(ii) Consider the q-focal BVP

$$Ly = y^{(n)} - \lambda p(x) y = f(x, y) \tag{3.1}$$

with boundary conditions

$$y^{(i)}(a) = c_i, \qquad i = 0, 1, \ldots, q-1,$$

$$y^{(j)}(b) = c_j, \qquad j = q, \ldots, n-1, \tag{3.2}$$

where $p(x) > 0$ is continuous, $\lambda = \pm 1$ and the equation $Ly = 0$ is disfocal on I, i.e., has no nontrivial solution $y(x)$ such that each of the derivatives $y^{(k)}(x)$, $k = 0, 1, \ldots, n-1$, vanishes at least once in I (see [3]). Peterson [5] has determined the Green's function for the associated homogeneous problem and has shown that its sign is determined by the inequality

$$(-1)^{n-q}G(x, s) > 0 \quad \text{for all } (x, s) \in (a, b) \times (a, b).$$

Hence in this case $I = I_1$ if $n - q$ is odd; $I = I_2$ if $n - q$ is even.

REMARKS. (i) Because of the general form of the boundary conditions (2.2), we require that the initial approximants v_1 and w_1 satisfy the boundary conditions. For the conjugate k-point BVP, this requirement can be relaxed somewhat. A modification of the iteration (2.7) or (2.14) then leads to the conclusions of Theorem 1 or Theorem 2, if one begins with functions v_1 and w_1 satisfying the boundary conditions (3.1)–(3.4) of Theorem 3.1 of [6].

(ii) Theorems 1 and 2 remain valid under Carathéodory conditions, in the case of Theorem 1 under the hypothesis that (2.4) holds for $(x, y) \in I \times \mathbf{R}$ for almost all x. Theorem 2 extends a result of V. Šeda [7] to the case of general linear boundary conditions.

(iii) For certain boundary problems, not only the sign of Green's function $G(x, s)$, but also the signs of some of its derivatives $\partial^{r_p}G(x, s)/\partial x^{r_p}$, $p = 1, \ldots, p_0$, $r_p < n$, are independent of s (see, for instance, [5]). (As an example, for the problem $y'' = 0$, $y(0) = a$, $y'(1) = b$, $G(x, s) \leqslant 0$, $\partial G(x, s)/\partial x \leqslant 0$.) In such cases, Theorems 1 and 2 can be extended in a natural way to problems of the form $Ly = f(x, y, y^{(r_1)}, \ldots, y^{(r_{p_0})})$, $Ty(x) = r$.

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