ON CYCLIC SUBGROUPS AND THE CONJUGACY PROBLEM

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ABSTRACT. The conjugacy problem in three types of group constructions involving cyclic subgroups is discussed. First it is shown that if G has the solvable conjugacy problem and if $h \in G$ and $k \in G$ satisfy (a) h and k are not power conjugate to themselves or each other, (b) the power conjugacy problem in G with respect to h or k is solvable, and (c) the double coset solvability problem in G is solvable with respect to $\langle h \rangle$ and $\langle k \rangle$, then the HNN extension $G^* = \langle G, t; t^{-1}ht = k \rangle$ has the solvable conjugacy problem. This result is used to deduce a similar theorem for free products with amalgamation, a fact first stated by Lipschutz. Then it is shown that if A and B are groups with the solvable conjugacy problem and $h \in A$ and $k \in B$ taken with themselves satisfy the conditions above in A and B , respectively, then $\langle A * B; [h, k] = 1 \rangle$ has the solvable conjugacy problem.

1. Introduction. The conjugacy problem in free products with cyclic amalgamated subgroups and in HNN extensions with cyclic associated subgroups has received a good deal of attention from various authors. Lipschutz [7] solved the conjugacy problem for the free product of free groups with cyclic amalgamated subgroups. Lipschutz [8] also proved that free products with cyclic amalgamations have the solvable conjugacy problem assuming that the factors have the solvable conjugacy problem and that the generators of the amalgamated cyclic subgroups are of a certain type, which he called semicritical.

Anshel and Stebe [2] first solved the conjugacy problem for an HNN extension of a free group with isomorphic cyclic associated subgroups. Anshel [1] extended this result to certain HNN groups having many stable letters.

Comerford and Truffault [4] solved the conjugacy problem in free products with cyclic amalgamations and HNN extensions with isomorphic cyclic associated subgroups where the factors or base groups, respectively, are small cancellation groups (specifically, sixth-groups). Their main technical result shows that elements of sixth-groups are semicritical for the free product with the amalgamation case and what we will call mutually semicritical in the HNN case.

Throughout this paper we will use the symbol $\tilde{H}$ for the conjugacy relation in the group H . When it is clear which group is meant, we will write $\sim$.

Following Lipschutz [8], we make the following definitions: Let G be a group, h and k elements of G . We will say that h and k are *mutually semicritical* if the following three conditions are satisfied:

(a) $h^n \sim h^m$ implies $n = m$, $k^n \sim k^m$ implies $n = m$, and $h \neq k$ implies $h^n \not\sim k^m$ for all integers m and n ,

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(b) for all $w \in G$, we can decide whether or not w is conjugate to a power of h , and the same for k ,

(c) for any $u \in G$, $v \in G$, we can decide whether or not there exist integers r and s such that $x'uy^s = v$ for all pairs $(x, y) \in \{h, k\} \times \{h, k\}$. We will say that h is *semicritical* if h is mutually semicritical with itself.

In this paper we will show:

THEOREM 1. *Let G have the solvable conjugacy problem and let h and k be mutually semicritical elements of G . Then the HNN extension $G^* = \langle G, t; t^{-1}ht = k \rangle$ has the solvable conjugacy problem.*

Then, using an argument from [5], we show how one may deduce Lipschutz's main result in [8] from Theorem 1. Finally, using the proof of Theorem 1 and the theory developed in [6], we show:

THEOREM 2. *Let A and B be groups with the solvable conjugacy problem, and let h and k be semicritical elements of A and B , respectively. Then the free product of A and B with commuting cyclic subgroups generated by h and k has the solvable conjugacy problem. We denote this group by $C = \langle A * B; [h, k] = 1 \rangle$.*

2. Proof of Theorem 1.

PROOF. Suppose u and v are words of G^* , $u \sim_G v$.

Case 1. Suppose u and v are t -free. Then by Collins' Lemma [3] there is a sequence $u = u_0, u_1, \dots, u_l = v$ such that $u_1, \dots, u_{l-1}$ are each in $\langle h \rangle$ or $\langle k \rangle$ and u_{i+1} is obtained from u_i by conjugating by an element of G into the cyclic associated subgroups and then by $t^{\pm 1}$ for $i = 0, \dots, l-1$.

If $u \sim_G h^\alpha$ for some α or $u \sim_G k^\beta$ for some β , or similarly for v , then $u \sim_{G^*} v$. We may decide this by (b). So suppose we determine that $u \sim_G x^\alpha$, $v \sim_G y^\beta$ for some $x, y \in \{h, k\}$ and $\alpha, \beta \in \mathbb{Z}$. The α and β are unique by (a). Conjugating by elements of G cannot change α and β because of (a), and conjugating by $t^{\pm 1}$ also keeps the powers fixed. So $u \sim_{G^*} v$ if and only if $\alpha = \beta$ and $x = y$.

Case 2. If either u or v has a t -symbol, they must be t -circumparallel by Collins' Lemma. We choose cyclic conjugates in $G * \langle t \rangle$ of form $u^* = u_1 t^{\epsilon_1} \dots u_n t^{\epsilon_n}$ and $v^* = v_1 t^{\epsilon_1} \dots v_n t^{\epsilon_n}$ where $u_i, v_i \in G$ and $\epsilon_i = \pm 1$ for $i = 1, \dots, n$. Of course u^* and v^* may effectively be determined from u and v .

As shown by Anshel and Stebe [2], $u \sim_{G^*} v$ if and only if there is an integral solution $\alpha_1, \dots, \alpha_n$ to the following system (*) of equations in G :

$$\begin{array}{rcl} u_1 y_1^{\alpha_1} & = & x_n^{\alpha_n} v_1 \\ u_2 y_2^{\alpha_2} & = & x_1^{\alpha_1} v_2 \\ \vdots & & \vdots \\ u_n y_n^{\alpha_n} & = & x_{n-1}^{\alpha_{n-1}} v_n \end{array} \quad \text{where } x_i = \begin{Bmatrix} h \\ k \end{Bmatrix} \text{ and } y_i = \begin{Bmatrix} k \\ h \end{Bmatrix} \text{ if } \epsilon_i = \begin{Bmatrix} -1 \\ +1 \end{Bmatrix}. \quad (*)$$

We will call an equation in (*) *mixed* if the x_i and y_j in that equation are different.

Suppose a mixed equation of (*) has a solution, say $u_i h^\alpha = k^\beta v_i$. If also $u_i h^\gamma = k^\delta v_i$, then $u_i^{-1} k^{\delta-\beta} u_i = h^{\gamma-\alpha}$, so we must have $\beta = \delta$ and $\gamma = \alpha$ by (a). Thus, a solution of a mixed equation of (*) is unique. Moreover, a fixed solution to one equation in (*) allows us effectively to determine if (*) has a solution, since we then successively have equations of form $x_i^\alpha = C$ where C is a fixed word of G , and these equations may be uniquely determined by (b) and (a). Thus, if (*) has a mixed equation, we may determine if this equation has a solution using (c), and from this we may proceed to determine if (*) has a solution.

The remaining case occurs when all equations in (*) are not mixed. Suppose without loss of generality that the first equation is of form $u_1 h^{\alpha_1} = h^{\alpha_n} v_1$. Then (*) must be of the following form:

$$\begin{array}{rcl} u_1 h^{\alpha_1} & = & h^{\alpha_n} v_1 \\ u_2 k^{\alpha_2} & = & k^{\alpha_n} v_2 \\ \vdots & & \vdots \\ u_{n-1} h^{\alpha_{n-1}} & = & h^{\alpha_n} v_{n-1} \\ u_n k^{\alpha_n} & = & k^{\alpha_n} v_n. \end{array}$$

Note that this may only occur if n is even and the t -projection of u and v alternating; otherwise (*) has a mixed equation.

Rewriting, we get

$$\begin{array}{rcl} h^{-\alpha_n} u_1 h^{\alpha_1} & = & v_1 \\ k^{-\alpha_n} u_2 k^{\alpha_2} & = & v_2 \\ \vdots & & \vdots \\ h^{-\alpha_n} u_{n-1} h^{\alpha_{n-1}} & = & v_{n-1} \\ k^{-\alpha_n} u_n k^{\alpha_n} & = & v_n. \end{array} \quad (**)$$

By (c) we may determine whether or not there exist integers p_i, q_i such that $x_i^{p_i} u_i x_i^{q_i} = v_i$ where x_i is the appropriate h or k . If for some i there are no such p_i and q_i , then (**) has no solution. So we now suppose we have found $p_1, \dots, p_n, q_1, \dots, q_n$ satisfying $x_i^{-p_i} u_i x_i^{q_i} = v_i$.

First of all we determine, by (c), if there are integers β_i, β'_i such that $x_i^{\beta_i} u_i x_i^{\beta'_i} = u_i$. If so, we must have $\beta'_i = -\beta_i$ by (a). So if there is such a β_i , let $x_i^{\gamma_i}$ be a generator of the subgroup of $\langle h \rangle$ or $\langle k \rangle$ commuting with u_i . Otherwise, let $\gamma_i = 0$. We may effectively determine γ_i for $i = 1, \dots, n$.

If $x_i^{-r_i} u_i x_i^{s_i} = v_i$ then $u_i^{-1} x_i^{r_i-p_i} u_i = x_i^{s_i-q_i}$, so $r_i - p_i = s_i - q_i$ by (a). This means $x_i^{r_i-p_i} \in \langle x_i^{\gamma_i} \rangle$. Thus we have $x_i^{-r_i} u_i x_i^{s_i} = v_i$ if and only if $r_i = p_i + m_i \gamma_i$ and $s_i = q_i + m_i \gamma_i$ for some integer m_i .

Note that if $\gamma_i = 0$ for some i , then the i th equation of (**) has a unique solution (p_i, q_i) , and we may proceed to determine whether or not (*) has a solution.

So suppose $\gamma_i \neq 0$ for $i = 1, \dots, n$. Then (*) has a solution $(\alpha_1, \dots, \alpha_n)$ if and only if the following system (***) of equations in $\mathbb{Z}$ has a solution $(m_1, \dots, m_n)$ in $\mathbb{Z}$:

$$\begin{array}{ccc}
q_1 + m_1\gamma_1 & = & p_2 + m_2\gamma_2 \\
q_2 + m_2\gamma_2 & = & p_3 + m_3\gamma_3 \\
\vdots & & \vdots \\
q_{n-1} + m_{n-1}\gamma_{n-1} & = & p_n + m_n\gamma_n \\
q_n + m_n\gamma_n & = & p_1 + m_1\gamma_1.
\end{array} \tag{***}$$

Solving these equations successively for m_i , we find that (***) has a solution if and only if $\sum_{j=1}^n p_j = \sum_{j=1}^n q_j$ and the following quantity is integral for $i = 2, \dots, n$:

$$m_i = \left(m_1\gamma_1 - \sum_{j=2}^i p_j - \sum_{j=1}^{i-1} q_j \right) / \gamma_i.$$

But $m_1\gamma_1 \equiv \sum_{j=2}^i p_j - \sum_{j=1}^{i-1} q_j \pmod{\gamma_i}$ for $i = 2, \dots, n$ if $m_1\gamma_1 \equiv \sum_{j=2}^i p_j - \sum_{j=1}^{i-1} q_j \pmod{\prod_{i=1}^n \gamma_i}$ for $i = 2, \dots, n$, so we need only check the elements in a complete residue system modulo $\prod_{i=1}^n \gamma_i$ to determine if the system of congruences has a solution. This completes the proof.

The following lemma establishes that the standard embedding of a free product with amalgamation into the corresponding HNN extension is conjugacy preserving.

LEMMA. *If A_i , $i \in I$, are groups with subgroups B_i , $i \in I$, and $\psi_i: B_i \rightarrow B_{i_0}$ are isomorphisms for $i \in I$, some fixed $i_0 \in I$, let $J = I - \{i_0\}$. Let H be the free product of the A_i , $i \in I$, amalgamating the subgroup B_i via the isomorphisms ψ_i . Let G^* be the analogously formed HNN extension of the free product P of the A_i , $i \in I$, with stable letters t_i , $i \in J$, that is,*

$$H = \langle P; b_i = \psi_i(b_i) \text{ for all } b_i \in B_i, (i \in J) \rangle,$$

$$G^* = \langle P, t_i, (i \in J); t_i^{-1}b_it_i = \psi_i(b_i) \text{ for all } b_i \in B_i, (i \in J) \rangle.$$

Let $\varphi: H \rightarrow G^*$ be the embedding defined by $\varphi(h) = h$ if $h \in A_{i_0}$, $\varphi(h) = t_i^{-1}ht_i$ if $h \in A_i$ for $i \in J$. Then φ has the property that, for all $h_1, h_2 \in H$, $h_1 \sim_H h_2$ if and only if $\varphi(h_1) \sim_{G^*} \varphi(h_2)$.

PROOF. The ‘only if’ part is trivial. We prove the ‘if’ part. Suppose h_1 and h_2 are reduced words of H and $\varphi(h_1) \sim_{G^*} \varphi(h_2)$. By Britton’s Lemma, since h_1 and h_2 are reduced in H (in the free product with amalgamation sense), $\varphi(h_1)$ and $\varphi(h_2)$ are reduced in G^* (in the HNN sense).

Claim. We may assume h_1 and h_2 are cyclically reduced in P .

PROOF OF CLAIM. If $\varphi(\tilde{h}_1) \sim_{G^*} \varphi(\tilde{h}_2)$ implies $\tilde{h}_1 \sim_H \tilde{h}_2$ for cyclically reduced $\tilde{h}_1$ and $\tilde{h}_2$, suppose $h_1 = c_1^{-1}\tilde{h}_1c_1$ in P and $h_2 = c_2^{-1}\tilde{h}_2c_2$ in P where $\tilde{h}_1$ and $\tilde{h}_2$ are cyclically reduced in P . Then $\varphi(h_1) \sim_{G^*} \varphi(h_2)$ implies

$$\varphi(c_1)^{-1}\varphi(\tilde{h}_1)\varphi(c_1) \sim_{G^*} \varphi(c_2)^{-1}\varphi(\tilde{h}_2)\varphi(c_2),$$

so $\varphi(\tilde{h}_1) \sim_{G^*} \varphi(\tilde{h}_2)$. This implies $\tilde{h}_1 \sim_H \tilde{h}_2$, and thus $h_1 \sim_H h_2$. The claim is proved.

Write $h_1 = p_1 \cdots p_n$ where $p_k \in A_{j_k}$ for some $j_k \in I$, $k = 1, \dots, n$, and $p_k \notin B_{j_k}$ for $k = 2, \dots, n$. Similarly write $h_2 = q_1 \cdots q_m$. Then $\varphi(h_1) = t_{j_1}^{-1}p_1t_{j_1} \cdots t_{j_n}^{-1}p_nt_{j_n}$ and $\varphi(h_2) = t_{i_1}^{-1}q_1t_{i_1} \cdots t_{i_m}^{-1}q_mt_{i_m}$ with $t_{i_0} = 1$.

Case 1. Suppose either $\varphi(h_1)$ or $\varphi(h_2)$ involve a t -symbol. By Collins' Lemma, $\varphi(h_1)$ and $\varphi(h_2)$ are t -circumparallel and $\varphi(h_1)^* = e^{-1}\varphi(h_2)^*e$ where $\varphi(h_1)^*$ and $\varphi(h_2)^*$ are cyclic conjugates of $\varphi(h_1)$ and $\varphi(h_2)$, respectively, both ending in $t_{i_d}^\varepsilon$ for some $i_d \in J$, $\varepsilon = \pm 1$. Also, $e \in A_{i_d}$ if $\varepsilon = -1$ and $e \in A_{i_0}$ if $\varepsilon = 1$. If $\varepsilon = 1$, we have in G^*

$$e^{-1}t_{j_k}^{-1}p_k t_{j_k} \cdots t_{j_{k-1}}^{-1}p_{k-1} t_{j_{k-1}} e = t_{l_r}^{-1}q_r t_{l_r} \cdots t_{l_{r-1}}^{-1}q_{r-1} t_{l_{r-1}}.$$

This is exactly $\varphi(e^{-1}p_k \cdots p_{k-1}e) = \varphi(q_r \cdots q_{r-1})$. Since φ is an embedding, we conclude $e^{-1}p_k \cdots p_{k-1}e = q_r \cdots q_{r-1}$ in H . This gives $p_k \cdots p_{k-1} \sim_H q_r \cdots q_{r-1}$, from which we have $h_1 \sim_H h_2$.

If $\varepsilon = -1$, we have in G^*

$$e^{-1}p_k t_{j_k} \cdots t_{j_{k-1}}^{-1}p_{k-1} t_{j_{k-1}} t_{j_k}^{-1} e = q_r t_{l_r} t_{l_{r+1}}^{-1} q_{r+1} t_{l_{r+1}} \cdots t_{l_{r-1}}^{-1} q_{r-1} t_{l_{r-1}} t_{l_r}^{-1}.$$

So $t_{i_d}^{-1}e^{-1}p_k t_{j_k} \cdots t_{j_{k-1}}^{-1}p_{k-1} t_{j_{k-1}} t_{j_k}^{-1} e t_{i_d} = t_{i_d}^{-1}q_r t_{l_r} \cdots t_{l_{r-1}}^{-1}q_{r-1} t_{l_{r-1}} t_{l_r}^{-1} t_{i_d}$. But $t_{i_d} = t_{j_k} = t_{l_r}$ and $e \in A_{i_d}$, $p_k \in A_{i_d}$, so this is exactly $\varphi(e^{-1}p_k \cdots p_{k-1}e) = \varphi(q_r \cdots q_{r-1})$, and we are done as in the case with $\varepsilon = 1$.

Case 2. Suppose neither $\varphi(h_1)$ nor $\varphi(h_2)$ involves a t -symbol. Then both $h_1 \in A_{i_0}$ and $h_2 \in A_{i_0}$, and $h_1 \sim_{G^*} h_2$. By Collins' Lemma either $h_1 \sim_P h_2$ and therefore $h_1 \sim_H h_2$, or there exists a sequence $u_1, \dots, u_s$ where $u_1 = h_1$, $u_s = h_2$ in P and either

(a) $u_{i+1} = x^{-1}u_i x$ where x is t -free and $u_{i+1} \in B_j$ for some j , or

(b) $u_{i+1} = t_j^{-\varepsilon} u_i t_j^\varepsilon$ where $\varepsilon = \pm 1$, and this a relation of G^* . We now show $u_i \sim_H u_{i+1}$ for $i = 1, \dots, s$.

(a) If $u_{i+1} = x^{-1}u_i x$ then $u_{i+1} \sim_P u_i$.

(b) If $\varepsilon = 1$ then $t_j^{-1}u_i t_j = u_{i+1}$ in G^* and $u_i \in B_j$ for some $j \in J$ with $u_{i+1} = \psi_i(u_i)$. Then $u_{i+1} = u_i$ in H .

If $\varepsilon = -1$ then $t_j u_i t_j^{-1} = u_{i+1}$ in G^* and $u_i \in B_{i_0}$ with $u_i = \psi_j(u_{i+1})$. Then $u_{i+1} = u_i$ in H . This completes the proof.

Using this lemma, we deduce Lipschutz' Theorem from [8].

THEOREM (LIPSCHUTZ). *Let H be the free product of groups with the solvable conjugacy problem amalgamating a cyclic subgroup generated by semicritical elements in the factors. Then H has solvable conjugacy problem.*

PROOF. We first consider the case $H = \langle A_1 * A_2; a_1 = a_2 \rangle$ where a_i is semicritical in A_i , $i = 1, 2$. Let $G^* = \langle A_1 * A_2, t; t^{-1}a_1 t = a_2 \rangle$. Since the embedding $\varphi: H \rightarrow G^*$ defined by $\varphi(g_2) = g_2$ if $g_2 \in A_2$, $\varphi(g_1) = t^{-1}g_1 t$ if $g_1 \in A_1$ is Frattini, we need only to solve the conjugacy problem in G^* . To do this, it suffices to show that a_1 and a_2 are mutually semicritical in $A_1 * A_2$.

(a) This is clear.

(b) If we cyclically reduce in the free product sense, this may be reduced to the conjugacy-power-solvability in the factors.

(c) Suppose we are given u and v in $A_1 * A_2$, and we seek a pair (r, s) so that $a_1^r u a_2^s = v$. We write $u = g_1 h_1 \cdots g_p h_p$ in normal form (free product) with perhaps $g_1 = 1$ or $h_p = 1$, and similarly $v = c_1 d_1 \cdots c_q d_q$, with $g_i, c_j \in A_1$, $h_i, d_j \in A_2$. We

consider the equation

$$d_q^{-1}c_q^{-1} \cdots d_1^{-1}c_1^{-1}a_1'g_1h_1 \cdots g_ph_pa_2^s = 1 \quad \text{in } A_1 * A_2.$$

We may determine by (a) in A_1 if there is an r with $a_1' \sim_A c_1g_1^{-1}$. If not, then there can be no pair (r, s) with $a_1'ua_2^s = v$. If so, r is unique by (a), so, using the word problem in A_1 we determine if $a_1' = c_1g_1^{-1}$. If this is true, we then determine if $d_1^{-1}h_1$ is trivial in A_2 , and continue. Finally, if we reach the word $d_q^{-1}h_pa_2^s$, we handle this exactly as $c_1^{-1}a_1'g_1$ was handled in the first step.

The other three cases to be considered in proving (c) are similar.

We now consider $H = \langle A_0 * \cdots * A_n; a_i = a_0, i = 1, \dots, n \rangle$ where a_i is semicritical in A_i for $i = 0, \dots, n$. Then the a_i are pairwise mutually semicritical in $P = A_0 * \cdots * A_n$. Let G^* be the group $G^* = \langle P, t_1, \dots, t_n; t_i^{-1}a_it_i = a_0, i = 1, \dots, n \rangle$. Then by the lemma the mapping $\varphi: H \rightarrow G^*$ with $\varphi(g_0) = g_0$ for $g_0 \in A_0$, $\varphi(g_i) = t_i^{-1}g_it_i$ for $g_i \in A_i, i = 1, \dots, n$, is a Frattini embedding. Thus, to solve the conjugacy problem for H , we need only solve the conjugacy problem for G^* . To do this, it suffices to show that G^* may be constructed as an HNN extension of the type considered in the first part of the theorem.

Let $G^0 = P$, $G^m = \langle G^{m-1}, t_m; t_m^{-1}a_mt_m = a_0 \rangle$ for $m = 1, \dots, n$. Then clearly $G^n = G^*$. To allow an iterated application of the theorem proved above for one stable letter, it remains to show that if $j > m + 1$ and a_0, a_j are mutually semicritical in G^m then they are mutually semicritical in G^{m+1} , $m = 0, \dots, n - 1$.

So suppose $j > m + 1$ and a_0, a_j are mutually semicritical in G^m . We check the three conditions in G^{m+1} .

(a) We know that a_0 and a_j are not self-power-conjugate or power-conjugate in G^m . If $a_0^\alpha \sim_{G^{m+1}} a_0^\beta$ then there is a sequence $a_0^\alpha = u_0, u_1, \dots, u_{l-1}, u_l = a_0^\beta$ where $u_1, \dots, u_{l-1}$ in $\langle a_0 \rangle$ and u_{i+1} is obtained from u_i by conjugating by an element of G^m into $\langle a_0 \rangle$ and then by $t_{m+1}^{\pm 1}$ for $i = 0, \dots, l - 1$. These conjugations cannot change the exponent of a_0 . For a_j the sequence must be completely trivial since $j > m + 1$, so $a_j^\alpha \sim_{G^{m+1}} a_j^\beta$ must imply $a_j^\alpha \sim_{G^m} a_j^\beta$. If $a_0^\alpha \sim_{G^{m+1}} a_j^\beta$ then a_j^β must enter into such a sequence starting with a_0^α , which is also impossible.

(b) We must be able to determine if there exists an α so that $w \sim_{G^{m+1}} a_j^\alpha$ and $w \not\sim_{G^m} a_j^\alpha$ for any w , and similarly for a_0 . For a_j this is similar to the argument in (a); since $j > m + 1$ we cannot have a_j^α in a Collins' Lemma type sequence. If $w \sim_{G^{m+1}} a_0^\alpha$ but $w \not\sim_{G^m} a_0^\alpha$, we must have $w \sim_{G^m} a_{m+1}^\alpha$, which may be decided (see Case 1 of Theorem 1).

(c) Given u and v in G^{m+1} we must be able to decide if, for example, there are integers r and s with $a_0^r u a_j^s = v$ in G^{m+1} . We may imitate the argument for (c) above in this theorem, using Britton's Lemma in the place of the normal form theorem for free products with amalgamation.

3. Proof of Theorem 2.

PROOF. We will assume that h and k have infinite order in A and B , respectively. If either h or k has finite order, the argument is similar. We will say that a word w of $P = A * B$ is *C-reduced* if it is reduced in P in the free product sense and has no

proper subword of the form hk or kh . A word w is *cyclically C-reduced* if every cyclic conjugate in the free product sense of w is C-reduced.

Suppose $u \sim v$ in C but $u \not\sim v$ in P . As shown in [6], there are two possible cases, assuming that u and v are cyclically C-reduced:

Case I. u and v have cyclic conjugates u^* and v^* such that $u^* = h^p k^q$ and $v^* = h^{p'} k^{q'}$ for some integers p, q, p', q' . Examining the labels of the only possible annular conjugacy diagram for u and v shows that the following equation holds in C for some integers $p_1, q_1, \dots, p_n, q_n$:

$$h^{p_1} k^{q_1} h^{p_2} k^{q_2} \dots h^{p_n} k^{q_n} h^{-p_{n-1}} k^{-q_{n-2}} \dots h^{-p_2} k^{-q_1} = 1$$

where $p_1 + p_2 = p$, $q_1 = q$, $-p_n = p'$, and $-(q_n + q_{n-1}) = q'$. But then we have $p_1 + \dots + p_n - p_{n-1} - \dots - p_3 = 0$ and $q_1 + \dots + q_n - q_{n-2} - \dots - q_2 = 0$, so $p_1 + p_2 = -p_n$ and $q_1 = -(q_{n-1} + q_n)$. Thus, $u^* = h^p k^q$ and $v^* = h^{p'} k^{q'}$ are conjugate if and only if $p = p'$ and $q = q'$.

Case II. u and v have cyclic conjugates u^* and v^* having one of the following forms:

(i) $u^* = u_1 k^{q_1} \dots u_n k^{q_n}$, $v^* = v_1 k^{q_1} \dots v_n k^{q_n}$ where $u_i, v_i \in A - \langle h \rangle$ and $q_1, \dots, q_n$ are nonzero integers, or

(ii) $u^* = u_1 h^{p_1} \dots u_n h^{p_n}$, $v^* = v_1 h^{p_1} \dots v_n h^{p_n}$ where $u_i, v_i \in B - \langle k \rangle$ and $p_1, \dots, p_n$ are nonzero integers.

We will consider words of form (i). Then $u \sim v$ if and only if the following system has a integral solution $(x_1, \dots, x_n)$:

$$\begin{array}{rcl} h^{x_1} u_2 \dots u_n h^{-x_n} & = & v_2 \dots v_n \\ h^{x_2} u_3 \dots u_1 h^{-x_1} & = & v_3 \dots v_1 \\ \vdots & & \vdots \\ h^{x_{n-1}} u_n \dots u_{n-2} h^{-x_{n-2}} & = & v_n \dots v_{n-2} \\ h^{x_n} u_1 \dots u_{n-1} h^{-x_{n-1}} & = & v_1 \dots v_{n-1}. \end{array}$$

We abbreviate $u_i \dots u_{i-2}$ and $v_i \dots v_{i-2}$ (modulo n , of course) by U_i and V_i , respectively. Thus, we have the system $h^{x_{i+1}} U_i h^{-x_i} = V_i$ for $i = 1, \dots, n$, where $x_{n+1} = x_1$. But the algorithm given in Theorem 1 to determine if system (**) has a solution $(\alpha_1, \dots, \alpha_n)$ can be used to determine if this system has a solution $(x_1, \dots, x_n)$.

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