A REMARK ON CARLESON'S CHARACTERIZATION OF BMO

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ABSTRACT. L. Carleson showed that if $\varphi \in \text{BMO}(R^n)$ and $\text{supp } \varphi$ is compact, then φ can be written in the form $\varphi(x) = \sum_{k=1}^{\infty} \int P_{t_k(y)}(x-y)b_k(y) dy + b_0(x)$ where $\sum_{k=0}^{\infty} \|b_k\|_{\infty} < C\|\varphi\|_{\text{BMO}}$, $t_k(y) > 0$ and $P_t(x) = C_n t(|x|^2 + t^2)^{-(n+1)/2}$ is the Poisson kernel. We show that we can take $b_2 = b_3 = \dots = 0$. This can be generalized on the space of homogeneous type with certain assumptions.

1. Introduction. All functions considered are real valued measurable functions. For $\varphi \in L^1_{\text{loc}}(R^n)$ we define

$$\|\varphi\|_{\text{BMO}} = \sup_B \inf_{c \in R} |B|^{-1} \int_B |\varphi(x) - c| dx$$

where the supremum is taken over all balls B and $|B|$ denotes the Lebesgue measure of B . And we define $\text{BMO}(R^n) = \{\varphi: \|\varphi\|_{\text{BMO}} < \infty\}$. In [3], Fefferman and Stein showed that $\text{BMO}(R^n)$ is the dual space of $H^1(R^n)$, where

$$\begin{aligned} \|g\|_{H^1} &= \|g\|_{L^1} + \sum_{j=1}^n \|R_j(g)\|_{L^1}, \\ R_j(g) &= F^{-1} \left(\xi_j \left(\sum_{k=1}^n \xi_k^2 \right)^{-1/2} (F(g))(\xi) \right) \end{aligned}$$

and F denotes the Fourier transform.

Recently L. Carleson [1] showed a new characterization of BMO which gives another proof of the duality between H^1 and BMO, that is,

THEOREM A. Let $h \in C^1(R^n)$ and assume that $h(x)$, $|\nabla h(x)| \leq (1 + |x|)^{-n-1}$, $h(x) \geq 0$, $\int h(x) dx = 1$ and $h(x)$ is even. Let $\|\varphi\|_{\text{BMO}} \leq 1$ and $\text{supp } \varphi \subset B_1$, where $B_t = \{x \in R^n: |x| < t\}$. Then there exists a positive number M , only depending on n , a sequence of functions $b_k(x)$ such that $\sum_{k=0}^{\infty} \|b_k\|_{\infty} \leq M$ and $t_k(y) > 0$ such that

$$\varphi(x) = \sum_{k \in N} \int h_{t_k(y)}(x-y)b_k(y) dy + b_0(x)$$

where $h_t(x) = t^{-n}h(x/t)$ and $N = \{1, 2, 3, \dots\}$.

Carleson's proof showed that we can take

$$b_2 = b_3 = \dots = 0 \tag{1}$$

Received by the editors November 7, 1978.

AMS (MOS) subject classifications (1970). Primary 46E99; Secondary 46E30.

Key words and phrases. BMO, Poisson kernel, space of homogeneous type.

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0002-9939/80/0000-0206/\$02.75

provided that $\text{supp } h \subset B_1$. But, in Fourier analysis, important cases occur when the support of h is not bounded, for example, the Poisson kernel

$$P_t(x) = C_n t / (|x|^2 + t^2)^{(n+1)/2}.$$

Now, we show (1) is possible in general.

THEOREM 1. *Let φ and h be as in Theorem A. Then there exists a positive number M , depending only on n , bounded functions $b_0(x)$, $b_1(x)$ and a positive function $t(y)$ such that*

$$\|b_0\|_\infty + \|b_1\|_\infty \leq M, \quad (2)$$

$$\varphi(x) = \int h_{t(y)}(x-y)b_1(y) dy + b_0(x). \quad (3)$$

Concerning the duality between H^1 and BMO, our modification does not give more information than does Carleson's result, but it means that there exists a Carleson measure ν of a special type such that

$$\varphi(x) = \int_{R^n \times (0, \infty)} P_t(x-y) d\nu(y, t) + b_0(x).$$

(For the definition of the Carleson measure see [6, p. 222].)

In §3, we extend this result to the space of homogeneous type with certain geometric assumptions, in particular to $\text{BMO}(\partial B^n)$.

The letters C and c will be used to denote various constants and the various uses of them do not all denote the same constant.

2. The main theory. In the proof of Theorem A, Carleson showed

THEOREM B. *Let φ and h be as in Theorem A. Then there exist functions $\{\beta_{1k}(x)\}_{k \in N}$, $\beta_1(x)$ and $\varphi^{(1)}(x)$ such that $\text{supp } \beta_{1k} \subset B_c$, $\text{supp } \beta_{1k} \cap \text{supp } \beta_{1h} = \emptyset$ ($k \neq h$), $\|\beta_{1k}\|_\infty < C$, $\|\beta_1\|_\infty < C$, $\text{supp } \varphi^{(1)} \subset B_c$, $\|\varphi^{(1)}\|_{\text{BMO}} < 1/2$, $r_k = C2^{-ck}$ and*

$$\varphi(x) = \sum_{k \in N} h_{r_k} * \beta_{1k}(x) + \beta_1(x) + \varphi^{(1)}(x).$$

Next he applied Theorem B to $2\varphi^{(1)}(2^{Ck_0}x)$, where k_0 is sufficiently large. Then

$$2\varphi^{(1)}(2^{Ck_0}x) = \sum_{k \in N} h_{r_k} * \beta'_{2k}(x) + \beta'_2(x) + \varphi^{(2)'}(x).$$

Changing the scale and multiplying by $\frac{1}{2}$,

$$\varphi^{(1)}(x) = \sum_{k \in Z, k > -k_0} h_{r_k} * \beta_{2k}(x) + \beta_2(x) + \varphi^{(2)}(x)$$

where $Z = \{0, \pm 1, \pm 2, \dots\}$. Repeating this argument, we get

$$\varphi(x) = \sum_{j \in N} \left(\sum_{k \in Z} h_{r_k} * \beta_{jk}(x) + \beta_j(x) \right)$$

where $\|\beta_j\|_\infty \leq 2^{-j+1}C$, $\|\beta_{jk}\|_\infty \leq 2^{-j+1}C$, $\text{supp } \beta_{jk} \subset B_c$, $\{\text{supp } \beta_{jk}\}_{k \in Z}$ are mutually disjoint, $\beta_{jk}(x) \equiv 0$ for $k \leq -jk_0$.

This implies Theorem A. In the above we note that

$$\left\| \sum_{j \in N} \sum_{k=-\infty}^0 h_{r_k} * |\beta_{jk}| \right\|_{\infty} < C,$$

$$\sum_{j \in N} \sum_{k \in N} h_{r_k} * |\beta_{jk}|(x) < \infty \quad \text{for a.e. } x.$$

Thus changing the order of the summation, we get

$$\varphi(x) = \sum_{k \in N} h_{r_k} * \left(\sum_{j \in N} \beta_{jk} \right)(x) + \sum_{j \in N} \sum_{k=-\infty}^0 h_{r_k} * \beta_{jk}(x) + \sum_{j \in N} \beta_j(x).$$

Since the last two terms are bounded, we get

COROLLARY. Assume that $\|\varphi\|_{\text{BMO}} < C$ and $\text{supp } \varphi \subset B_1$. Then there exist bounded functions $\{d_k(x)\}_{k \in N}$, $d(x)$ and positive numbers $\{r_k\}_{k \in N}$ such that

$$\left\| \sum_{k \in N} |d_k| \right\|_{\infty} < 1, \quad (4)$$

$$\|d\|_{\infty} < 1, \quad (5)$$

$$\varphi(x) = \sum_{k \in N} h_{r_k} * d_k(x) + d(x). \quad (6)$$

To prove Theorem 1, we need the above corollary and the following

LEMMA. Let (Ω, F, μ) be a σ -finite measure space with no atoms. Let $\{F_k\}_{k \in N}$ be a sequence of nondecreasing atomic sub- σ -fields of F . Let A_k be the set of the atoms of F_k . Let $\{u_k(x)\}_{k \in N}$ be a sequence of nonnegative functions such that $\|\sum_{k \in N} u_k\|_{\infty} < 1$. Then there exist mutually disjoint F -measurable sets $\{H_k\}_{k \in N}$ such that

$$\mu(H_k \cap B) = \int_B u_k(x) d\mu(x) \quad (7)$$

for any $k \in N$ and $B \in A_k$.

PROOF OF THEOREM 1. For simplicity we consider the case $n = 1$. We may assume that $\|\varphi\|_{\text{BMO}}$ is sufficiently small. By the corollary, we get bounded functions $\{d_k(x)\}_{k \in N}$, $d(x)$ and positive numbers $\{r_k\}_{k \in N}$ with (4), (5) and (6). Let $\{\varepsilon_k\}_{k \in N}$ be positive numbers such that $2^{-k}r_k > \varepsilon_k$ and $\varepsilon_k/\varepsilon_{k+1} \in N$ for any $k \in N$. Let $\{F_k\}_{k \in N}$ be the atomic sub- σ -fields such that $A_k = \{[j\varepsilon_k, (j+1)\varepsilon_k) : j \in \mathbb{Z}\}$, where A_k is as in the lemma. Set $u_k(x) = |d_k(x)|$. Then, by (4) and the lemma, we get mutually disjoint measurable sets $\{H_k\}_{k \in N}$ with (7). If $x \in H_k$, we put

$$f_k(x) = |H_k \cap [j\varepsilon_k, (j+1)\varepsilon_k)|^{-1} \int_{j\varepsilon_k}^{(j+1)\varepsilon_k} d_k(y) dy$$

where $x \in [j\varepsilon_k, (j+1)\varepsilon_k)$. If $x \notin H_k$, we put $f_k(x) = 0$. Then, by (7), $\|f_k\|_{\infty} < 1$.

And

$$\begin{aligned}
 & \left| \int_R h_{r_k}(x-y)(d_k(y) - f_k(y)) dy \right| \\
 & \leq \sum_{j \in Z} \int_{j\epsilon_k}^{(j+1)\epsilon_k} |h_{r_k}(x-y) - h_{r_k}(x-j\epsilon_k)| |d_k(y) - f_k(y)| dy \\
 & \leq (\|d_k\|_\infty + \|f_k\|_\infty) C(\epsilon_k/r_k) \int_R (1 + |x-y|/r_k)^{-2} r_k^{-1} dy < C 2^{-k}.
 \end{aligned}$$

Thus, by (6), $\varphi(x) = \sum_{k \in N} h_{r_k} * f_k(x) + \sum_{k \in N} h_{r_k} * (d_k - f_k)(x) + d(x)$ where $\|\sum_{k \in N} h_{r_k} * (d_k - f_k) + d\|_\infty < C$, $\|f_k\| < 1$ and $\{\text{supp } f_k\}_{k \in N}$ are mutually disjoint. Set $b_1(x) = f_k(x)$ on H_k for $k \in N$, $b_1(x) = 0$ on $(\cup H_k)^c$, $b_0(x) = \sum_{k \in N} h_{r_k} * (d_k - f_k)(x) + d(x)$, $t(y) = r_k$ on H_k for $k \in N$, $t(y) = 1$ on $(\cup H_k)^c$. Then the conditions (2) and (3) are satisfied. Q.E.D.

3. BMO on the space of homogeneous type. Following [2], we consider BMO on the space of homogeneous type. Let X be a topological space with a Borel measure μ , a quasi-distance ρ and functions $\{K_r\}_{0 < r < \mu(X)}$ on $X \times X$ satisfying the following properties: there exist positive numbers C_1, C_2 and α such that

- (a) $\rho(x, y) = \rho(y, x) \geq 0$,
- (b) $\rho(x, y) = 0$ iff $x = y$,
- (c) $\rho(x, z) \leq C_1(\rho(x, y) + \rho(y, z))$,
- (d) $C_2 r \leq \mu(B_r(x)) \leq C_1 r$ where $B_r(x) = \{y \in X: \rho(x, y) < r\}$ and $0 < r < \mu(X)$,
- (e) $\{B_r(x)\}_{r > 0}$ form a basis of open neighbourhoods of x ,
- (f) $|\rho(x, y) - \rho(x, z)|/\rho(x, y) \leq C_1(\rho(y, z)/\rho(x, y))^\alpha$ if $\rho(y, z) < C_2 \rho(x, y)$,
- (g) $K_r(x, y) \geq 0$,
- (h) $\inf\{\int_{B_r(z)} K_{r/C_1}(x, y) d\mu(y) : r > 0, z \in X, B_r(z) \ni x\} > C_2$,
- (i) $\sup_{x, r} |\int_X K_r(x, y) d\mu(y) - 1| < 1/10$,
- (j) $K_r(x, y) \leq C_1(1 + \rho(x, y)/r)^{-1-\alpha}/r$,
- (k) $\max(|K_r(x, y) - K_r(z, y)|, |K_r(y, x) - K_r(y, z)|) \leq C_1(\rho(x, z)/r)^\alpha(1 + \rho(x, y)/r)^{-1-\alpha}/r$ if $\rho(x, z) \leq C_2 r$.

For $f \in L^1_{\text{loc}}(X)$ (i.e. $f|_B \in L^1(B)$ for all $B = B_r(x)$) we define

$$\|f\|_{\text{BMO}(X)} = \sup_B \inf_{c \in R} |B|^{-1} \int_B |f(x) - c| d\mu(x)$$

if $\mu(X) = \infty$, where the supremum is taken over all $B = B_r(x)$ ($r > 0, x \in X$). If $\mu(X) < \infty$, we define

$$\|f\|_{\text{BMO}(X)} = \sup_B \inf_{c \in R} |B|^{-1} \int_B |f(x) - c| d\mu(x) + \left| \int_X f(x) d\mu(x) \right|.$$

Then we can extend Theorem 1 to this space.

THEOREM 2. *Let $\|\varphi\|_{\text{BMO}(X)} < 1$ and $\text{supp } \varphi \subset B_{r_0}(x_0)$ for some $r_0 > 0$ and $x_0 \in X$. Then there exist bounded functions $b_0(x)$ and $b_1(x)$ such that $\|b_0\|_\infty + \|b_1\|_\infty \leq M$ and a positive function $t(y)$ such that the range of t is a discrete subset of $R^+ \setminus \{0\}$*

and that

$$\varphi(x) = \int K_{r(y)}(x, y) b_1(y) d\mu(y) + b_0(x)$$

where M is independent of φ , r_0 and x_0 .

The proof is the same as with Theorem 1 and we will omit it. [This kind of generalization is announced in [2] under the assumption $K_r(x, y) = 0$ if $\rho(x, y) > r$.]

EXAMPLE. ∂B^n denotes $\{z = (z_1, \dots, z_n) \in C^n: \sum_{k=1}^n |z_k|^2 = 1\}$. On ∂B^n we define $\rho(z, w) = |1 - \sum_{k=1}^n z_k \bar{w}_k|^n$ and $K_r(z, w) = P((1 - r^{1/n})z, w)$, where $P(\xi, z) = C_n(1 - |\xi|^2)^n / |1 - \xi \cdot \bar{z}|^{2n}$ that is the Poisson-Szegő kernel. Then, by using the Lebesgue surface measure $d\sigma$, we obtain a space of homogeneous type with (a)–(k). Thus, by Theorem 2, for any $\varphi \in \text{BMO}(\partial B^n)$, there exist bounded functions $b_0(z)$, $b_1(z)$ and $t(z)$ such that $1/2 < t(z) < 1$, $\|b_0\|_\infty + \|b_1\|_\infty \leq M\|\varphi\|_{\text{BMO}}$ and

$$\varphi(z) = \int_{\partial B^n} P(t(w)w, z) b_1(w) d\sigma(w) + b_0(z).$$

4. Proof of lemma. In the following $k, p, q \in N$. We may assume

$$\sup \left\{ \mu(B): B \in \bigcup_{k \in N} A_k \right\} < \infty, \\ \lim_{k \rightarrow \infty} \sup \{ \mu(B): B \in A_k \} = 0. \quad (8)$$

Let $A_1 = \{B(1), B(2), \dots\}$. For each $B(p)$ let $\{B(p, 1), B(p, 2), \dots\}$ be the elements of A_2 contained in $B(p)$. In this way A_k ($k \in N$) can be indexed inductively as follows.

$$A_k = \{B(p_1, \dots, p_k)\}_{1 \leq p_q < N_q(p_1, \dots, p_{q-1})} \quad (q = 1, \dots, k),$$

$$N_q(p_1, \dots, p_{q-1}) \in \{2, 3, 4, \dots, \infty\}, \quad B(p_1, \dots, p_k) \subset B(p_1, \dots, p_q)$$

$$\text{if } q \in \{1, \dots, k\}.$$

Set $I = \{(p_1, \dots, p_k): k \in N, 1 \leq p_q < N_q(p_1, \dots, p_{q-1}) \text{ for } q = 1, \dots, k, |p_1, \dots, p_k| = k \text{ and } (p_1, \dots, p_k)(q) = p_q \text{ for } q = 1, 2, \dots, k\}$. In the following (i) and (j) stand for elements of I . (i) < (j) means $|i| > |j|$ and (i)(k) = (j)(k) for $k = 1, \dots, |j|$.

For each (i) $\in I$ and $a \in [0, \mu(B(i)) - \int_{B(i)} \sum_{k=|i|+1}^\infty u_k d\mu]$, we set $m = |i|$,

$$M_{m+1}((i), a) = \inf \left\{ p: \sum_{k=1}^p \left(\mu(B(i, k)) - \int_{B(i, k)} \sum_{q=m+1}^\infty u_q d\mu \right) > a \right\},$$

$$a_{m+1}((i), a) = a - \sum_{k=1}^{M_{m+1}-1} \left(\mu(B(i, k)) - \int_{B(i, k)} \sum_{q=m+1}^\infty u_q d\mu \right)$$

$$+ \int_{B(i, M_{m+1})} u_{m+1} d\mu,$$

$$G_{m+1}((i), a) = \bigcup_{k=1}^{M_{m+1}-1} B(i, k).$$

By the condition on a , M_{m+1} is well defined. Since, by the definition of M_{m+1} ,

$$a_{m+1} \in \left[0, \mu(B(i, M_{m+1})) - \int_{B(i, M_{m+1})} \sum_{k=m+2}^{\infty} u_k d\mu \right),$$

we can define M_{m+k} , a_{m+k} and G_{m+k} inductively as follows:

$$M_{m+k}((i), a) = M_{m+k}((i, M_{m+1}), a_{m+1}((i), a)),$$

$$a_{m+k}((i), a) = a_{m+k}((i, M_{m+1}), a_{m+1}((i), a)),$$

$$G_{m+k}((i), a) = G_{m+k}((i, M_{m+1}), a_{m+1}((i), a)).$$

Set

$$E((i), a) = \bigcup_{k=1}^{\infty} G_{|i|+k}((i), a),$$

$$F(i) = E\left((i), \int_{B(i)} u_{|i|} d\mu\right),$$

$$J_k(i) = \{B \in A_k : \mu(B \cap F(i)) > 0\} \quad \text{for } k > |i|.$$

Then

$$(9) F(i) \subset B(i).$$

$$(10) \mu(F(i)) = \sum_{k=|i|}^{\infty} \sum_{B \in J_k(i)} \int_B u_k d\mu.$$

$$(11) \text{ If } \mu(F(i) \cap B(j)) > 0 \text{ and } (i) > (j), \text{ then } F(i) \supset F(j).$$

$$(12) \text{ If } \mu(F(i) \cap F(j)) = 0, \text{ then } J_k(i) \cap J_k(j) = \emptyset \text{ for any } k > \max(|i|, |j|).$$

$$(13) \text{ If } \int_{B(i)} u_{|i|} d\mu > 0, \text{ then } \mu(F(i)) > 0.$$

Set $K(i) = F(i) \setminus \bigcup_{(j) < (i)} F(j)$. By (9) and (11), there exists a subset $L(i)$ of I with the properties: (a) $(j) < (i)$ and $F(j) \subset F(i)$ for any $(j) \in L(i)$, (b) $\{F(j)\}_{(j) \in L(i)}$ are mutually disjoint, (c) if $(j) < (i)$ and $F(j) \subset F(i)$, then there exists $(j_0) \in L(i)$ such that $F(j) \subset F(j_0)$. Thus by (10)–(12) and the above property of $L(i)$, we have

$$\begin{aligned} \mu(K(i)) &= \mu(F(i)) - \sum_{(j) \in L(i)} \mu(F(j)) \\ &= \sum_{k=|i|}^{\infty} \sum_{B \in J_k(i)} \int_B u_k d\mu - \sum_{(j) \in L(i)} \sum_{k=|j|}^{\infty} \sum_{B \in J_k(j)} \int_B u_k d\mu \\ &= \int_{B(i)} u_{|i|} d\mu. \end{aligned}$$

Set $H_k = \bigcup_{(i) \in I, |i|=k} K(i)$, then $\{H_k\}_{k=1}^{\infty}$ has the required property.

Lastly we prove (9)–(13). (9) and (10) are obvious. (13) follows from the assumption (8). Note that

$$E((i), a_1) \subset E((i), a_2) \tag{14}$$

provided $0 \leq a_1 \leq a_2 < \mu(B(i)) - \int_{B(i)} \sum_{k=|i|+1}^{\infty} u_k d\mu$ and note that

$$\begin{aligned} E((i), a) &= \left(\bigcup_{k=|i|+1}^p G_k((i), a) \right) \cup \left(\bigcup_{k=p+1}^{\infty} G_k((i), a) \right) \\ &\supset \bigcup_{k=p+1}^{\infty} G_k((i), a) = E((i, M_{|i|+1}((i), a), \dots, M_p((i), a)), a_p((i), a)) \end{aligned} \quad (15)$$

provided $p > |i|$. From (14), (15) and

$$a_p((i), a) \geq \int_{B(i, M_{|i|+1}((i), a), \dots, M_p((i), a))} u_p d\mu,$$

(11) and (12) follow easily. Q.E.D.

ACKNOWLEDGEMENT. I would like to thank Professor Richard R. Goldberg for valuable information.

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