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ON AUTONOMOUS CONTROL SYSTEMS ON CERTAIN MANIFOLDS[1]

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ABSTRACT. Let M^n be a compact C^∞ manifold, $n > 4$, admitting a vector field with every orbit a circle. Then there exists a completely controllable set S consisting of two nonsingular C^∞ vectors X and Y such that every orbit of X is a circle.

An autonomous control system on a smooth manifold M is the same as a set of vector fields on M. A set S of vector fields on a smooth manifold M is said to be controllable if for every pair (m, m') of points of M there exists a trajectory of S from m to m'. Here a trajectory of S is a curve which is an integral curve (orbit) of some $X \in S$ or a finite concatenation of such curves such that a trajectory of S run in reverse is not allowed. (We refer the readers to **[2]** for details.)

In **[2]**, N. Levitt and H. J. Sussmann showed that on every connected paracompact manifold of class C^k, $2 \leqslant k \leqslant \infty$, or $k = \omega$, there exists a completely controllable set S consisting of two vector fields of class C^{k-1}.

For simplicity, we assume that all the manifolds, vector fields, etc., considered here are of class C^∞.

A manifold M is called closed if it is compact and without boundary. Let D^k denote the k-dimensional disk and S^{k-1} its boundary.

The purpose of this paper is to prove the following theorem:

THEOREM. *If a connected closed n-dimensional manifold M^n, $n > 4$, admits a vector field X_0 with every orbit a (nondegenerate) circle, then there exists a completely controllable set S consisting of two nonsingular vectors X and Y such that every orbit of X is a circle and Y has finitely many closed orbits.*

We first give a brief sketch of the proof.

According to **[1]**, M can be decomposed as a union of round handles $R_k = S^1 \times D^k \times D^{n-k-1}$. Each round k-handle R_k is supplied with a vector field

$$V = d/dt - \sum x_i \partial/\partial x_i + \sum y_j \partial/\partial y_j,$$

where $(t, x, y) \in S^1 \times D^k \times D^{n-k-1}$. A point $p \in R_k$ can be moved along a trajectory of V arbitrarily close to the closed orbit $S^1 \times 0 \times 0$ if and only if $p \in S^1 \times D^k \times 0$. Modifying the vector field V on each R_k, we will get a

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nonsingular vector field W such that for $k > 0$ (respectively $k = 0$) any trajectory of W approaching $S^1 \times 0 \times 0$ (respectively any trajectory of W) meets $S^1 \times$ (x_1-axis) (respectively $S^1 \times$ (y_1-axis)) in R_k. By patching up the vector fields W on the R_k's as in [1], we construct a nonsingular vector field Y on M with finitely many closed orbits $\{C_i\}$, where C_i corresponds to $S^1 \times 0 \times 0$ on each R_k. By using the standard transversality argument, we may assume that near C_i for each $t \in S^1$ the x_1-axis (or y_1-axis) forms part of an orbit of X. Then we construct a sequence of diffeomorphisms $f_1, \ldots, f_{N-1}$ of M to itself such that the C_i's are connected by the orbits of the vector field $X = f_{(N-1)*} \cdots f_{1*}(X_0)$. The set $\{X, Y\}$ is showed to be completely controllable.

Let V denote the vector

$$d/dt - \sum_{i=1}^{k} x_i \partial/\partial x_i + \sum_{j=1}^{n-k-1} y_j \partial/\partial y_j$$

on $R_k = S^1 \times D^k \times D^{n-k-1}$, where the x_i's and y_j's denote the standard coordinate functions on R^k and R^{n-k-1} respectively.

LEMMA 1. *For $k > 0$, there exists a vector field W on R_k such that W has $S^1 = S^1 \times 0 \times 0$ as its only closed orbit, and $S^1 \times (x_1\text{-axis} - 0)$ is reachable from every trajectory except S^1.*

PROOF. Assume that $k > 1$, let $B_j \subseteq D^k$ be the 2-dimensional disk spanned by x_1-axis and x_j-axis, $1 < j \leqslant k$. We write rB_j for the concentric disk of radius r. For small $\theta_0 > 0$, we may construct a diffeomorphism $g_{\theta,j}\colon B_j \to B_j$ for each $0 < \theta < \theta_0$ such that $g_{\theta,j}$ fixes the complement of $\frac{3}{4}B_j$, $g_{\theta,j}|\frac{1}{4}B_j =$ the rotation by a degree of θ, and $G_j(\theta, x_1, x_j) = g_{\theta,j}(x_1, x_j)$, $0 < \theta < \theta_0$, is an isotopy with $g_0 =$ identity. Then we define $g_{\theta,j}$ for arbitrary $\theta > 0$ by $g_{\theta,j} = g_{r,j} g^p_{\theta_0,j}$, where $\theta = p\theta_0 + r$ with p an integer and $r > 0$. The diffeomorphism $g_{\theta,j}$ induces a diffeomorphism $h_{\theta,j}$ on $D^k \times D^{n-k-1}$ by fixing the remaining coordinates. We also write

$$H_j(\theta, x_1, \ldots, x_k, y_1, \ldots, y_{n-k-1}) = h_{\theta,j}(x_1, \ldots, y_{n-k-1}).$$

We construct an isotopy

$$F_j\colon [4j\pi, 4(j+1)\pi] \times D^k \times D^{n-k-1} \to D^k \times D^{n-k-1}$$

as follows:

$$F_j(t, x, y) = \begin{cases} H_j(t - 4j\pi, x, y) & \text{for } t \in [4j\pi, (4j+1)\pi], \\ H_j(\pi, x, y) & \text{for } t \in [(4j+1)\pi, (4j+2)\pi], \\ H_j((4j+3)\pi - t, x, y) & \text{for } t \in [(4j+2)\pi, (4j+3)\pi], \\ H_j(0, x, y) & \text{for } t \in [(4j+3)\pi, (4j+4)\pi]. \end{cases}$$

The map F_j induces a diffeomorphism K_j from $[4j\pi, 4(j+1)\pi] \times D^k \times D^{n-k-1}$ to itself, where $K_j(t, x, y) = (t, F_j(t, x, y))$. By gluing together K_j, $2 \leqslant j \leqslant k$, on the common boundaries, we obtain a diffeomorphism K of $[8\pi, 4n\pi] \times D^k \times D^{n-k-1}$ to itself. Identifying 8π with $4n\pi$, we thus have a diffeomorphism $\bar{K}$ of $S^1 \times D^k \times D^{n-k-1}$ (geometrically, $\bar{K}$ is given by twisting $\frac{1}{4}B_j$ along S^1 by 180°, and then twisting it back 180° successively for $2 \leqslant j \leqslant n-1$). We define W to be $\bar{K}_*(V)$ on R_k. Q.E.D.

The same proof yields the following lemma:

LEMMA 2. *There exists a nonsingular vector field W on $R_0 = S^1 \times D^0 \times D^{n-1} = S^1 \times D^{n-1}$ with every trajectory except S^1 leaving R_0, and every trajectory meets $S^1 \times (y_1$-axis).*

For a round k-handle $R_k = S^1 \times D^k \times D^{n-k-1}$, we write $\partial_- R_k = S^1 \times S^{k-1} \times D^{n-k-1}$, and $\partial_+ R_k = S^1 \times D^k \times S^{n-k-2}$ [**1**, p. 42].

PROOF OF THE THEOREM. Since M supports a nonsingular vector field X_0, its Euler number vanishes. According to [**1**], for $n \geqslant 4$, M admits a round handle decomposition, that is, M can be written as $R_0^1 + \cdots + R_0^{b_1} + \cdots + R_{n-1}^1 + \cdots + R_{n-1}^{b_{n-1}}$, where each R_k^i denote a round k-handle attached to the boundary of the stuff on the left to it, successively (using $\partial_- R_k^i$ as the attaching region at each stage), [**1**, p. 43]. Near $\partial_- R_k^i$, when $k > 0$, W points inwards (into R_k^i). Hence we may use the argument in [**1**, pp. 52–53] to patch up the W's to construct a vector field Y on M with finitely many closed orbits $\{C_k^j\}$, corresponding to the core $S^1 = S^1 \times 0 \times 0$ in R_k^j. Furthermore, by the standard transversality argument, we may assume that the orbits of X_0 meet each C_k^j transversely. Therefore, near each $C_k^j = S^1$, for each $t \in S^1$ the x_1-axis (or y_1-axis when $k = 0$) forms part of an orbit of X_0.

Recall that $M = R_0^1 + \cdots + R_0^{b_1} + \cdots + R_{n-1}^{b_{n-1}}$ with $C_k^j \subseteq R_k^j$. We order the C_k^j's from left to right in this decomposition, and denote them by $\{C_j\}_{j=0}^N$.

Now we are going to construct a sequence of diffeomorphisms f_j, $0 \leqslant j \leqslant N-1$, from M to itself such that C_j and C_{j+1} are connected by an orbit γ_j of $f_{j*}(X_j) = X_{j+1}$, and $f_j(\gamma_i) = \gamma_i$ when $i < j$. Let β_j be an orbit of X_j meeting C_j, and p a point on β_j but not on C_j (such a point exists, because of the transversality). We embed a curve ∂: $[0, 1] \to M$ with $\partial(0) = p$ and $\partial(1) = p'$, a point on C_{j+1}. Since $n \geqslant 4$, we may assume that $\partial((0, 1))$ does not intersect any of the other C_t's and γ_i's with $i < j$. Let U_j be a tubular neighborhood of $\partial([0, 1])$ in M with $U_j \cap C_{j+1} = l$, a line segment, and U_j is disjoint from all the other C_t's and γ_i's with $i < j$. As in [**1**, pp. 44–45], we construct an isotopy F_j with support in U_j from the identity to a diffeomorphism f_j with $f_j(p) = p'$ (geometricaly, we drag p to p' along the path ∂). The orbit $\gamma_j = f_j(\beta_j)$ of the vector field $f_{j*}(X_j) = X_{j+1}$ connects C_j and C_{j+1}. We apply this argument successively to get a vector field X_N with an orbit γ_j connecting C_j and C_{j+1} for each j with $0 < j < N-1$.

By using the transversality argument again if necessary, we perturb X_N near each C_k^j to get a vector field X (also with every orbit a circle) such that near C_k^j for each $t \in S^1$ the x_1-axis (or y_1-axis when $k = 0$) forms parts of an orbit of S.

We claim that $\{X, Y\}$ forms a completely controllable system. Given any two points m, m' on M. From the description of Y, we see that m' must be on a trajectory coming out from some R_0^i. On the other hand, m is on a trajectory approaching some $C_k^j \subseteq R_k^j$ with $k > 0$. We can reach C_r^i from C_k^j by a sequence of the trajectories γ_j of X constructed above and some closed orbits of Y. From Lemma 1, Lemma 2, and the last paragraph, we see that C_k^j is reachable from m, and m' is reachable from C_0^i. Hence m' is reachable from m by trajectories (in the positive direction) of the system $\{X, Y\}$. Q.E.D.

Any closed connected manifold M^n, $n \geqslant 4$, which is the total space of an S^1-bundle satisfies the conclusion of the theorem. For example, the odd-dimensional sphere S^{2k+1} is the total space of the Hopf bundle $S^1 \to S^{2k+1} \to \mathrm{CP}(k)$, where $\mathrm{CP}(k)$ denotes the k-dimensional complex projective space.

For $n = 3$, the proof shows that if M^3 satisfies the additional condition that it admits a round handle decomposition, then the conclusion also holds. For example, S^3 can be written as $S^1 \times D^2 + D^2 \times S^1$, one round 0-handle and one round 2-handle.

References

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