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ISOMETRIC IMMERSIONS OF COMPLETE RIEMANNIAN MANIFOLDS INTO EUCLIDEAN SPACE

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ABSTRACT. Let M be a complete Riemannian manifold of dimension n, with scalar curvature bounded from below. If the isometric immersion of M into euclidean space of dimension $n + q$, $q < n - 1$, is included in a ball of radius λ, then the sectional curvature K of M satisfies $\limsup_M K > \lambda^{-2}$. The special case where M is compact is due to Jacobowitz.

Generalizing results by Tompkins, Chern and Kuiper, and Otsuki, Jacobowitz proved that a compact n-dimensional Riemannian manifold whose sectional curvatures are everywhere less than constant λ^{-2} cannot be isometrically immersed into euclidean space of dimension $2n - 1$ so as to be contained in a ball of radius λ (see [1] and the references therein). In this note we shall prove a quantitative result concerning isometric immersions, which includes Jacobowitz's theorem as a special case.

The proof of our result will consist in a simple application of a theorem by Omori [3], which we now formulate.

Let M be a complete Riemannian manifold with sectional curvature bounded from below; consider a smooth function $f\colon M \to R$ with $\sup f < \infty$. For any $\varepsilon > 0$ there exists a point $p \in M$ where $\|\operatorname{grad} f\| < \varepsilon$ and $\nabla^2 f(X, X) < \varepsilon$ for all unit vectors $X \in T_pM$. By $\nabla^2 f$ we mean the Hessian form of f, defined by $\nabla^2 f(X, Y) = \langle \nabla_X \operatorname{grad} f, Y \rangle$.

THEOREM 1. *Let M be a complete n-dimensional Riemannian manifold with scalar curvature R bounded from below. Assume that there exists an isometric immersion φ of M into euclidean space of dimension $n + q$, $q \leq n - 1$, so that $\varphi(M)$ is included in a ball of radius λ. Then* $\limsup_M K \geq \lambda^{-2}$, *where K is the sectional curvature of M.*

COROLLARY. *A complete two-dimensional Riemannian manifold, immersed isometrically into euclidean three-space, and whose Gaussian curvature K satisfies $-\infty < -a^2 \leq K \leq 0$, is extrinsically unbounded.*

PROOF OF THE THEOREM. If $n = 2$ then $R = 2K$ and we have $\inf K > -\infty$. If $n > 2$ and $\inf K = -\infty$, then $\inf R > -\infty$ easily implies $\sup K = +\infty$ and the theorem follows. We may therefore assume $\inf K > -\infty$.

We shall apply Omori's theorem to the "distance" function $F = \langle \varphi, \varphi \rangle / 2$; φ is considered here as tangent vector in euclidean space E^{n+q}. By assumption, we have

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$\|\varphi\| \leqslant \lambda$ and $f \leqslant \lambda^2/2$, taking the origin to be the center of the ball wherein $\varphi(M)$ lies. Therefore, to any natural number m, there exists a point $p_m \in M$ where $\nabla^2 f(X, X) < 1/m$ for all $X \in Tp_m M$ with $\|X\| = 1$. In order to compute the Hessian of f, we identify every tangent vector X with $\varphi_*(X)$ and obtain $\nabla'_X \varphi = X$, where ∇' denotes the connection of E^{n+q}. Now using this and the Gauss formula, we compute easily $\nabla^2 f(X, Y) = \langle X, Y\rangle + \langle L(X, Y), \varphi\rangle$, where L stands for the second fundamental form of the immersion. Thus at p_m and for every nonzero $X \in Tp_m M$ we have $1 + \langle L(X, X), \varphi\rangle \cdot \|X\|^{-2} < m^{-1}$, hence

$$\lambda^{-1}(1 - m^{-1}) < \|L(X, X)\| \cdot \|X\|^{-2}. \tag{$*$}$$

From ($*$) we conclude that, at $p_m \in M$, we have $L(X, X) \neq 0$ for $X \neq 0$. Now we use, as in [1], a well-known algebraic lemma [2, p. 28]. Let $L: R^n \times R^n \to R^q$ be symmetric, bilinear and satisfy $L(X, X) \neq 0$ for $X \neq 0$; if $q \leqslant n - 1$, there exist linearly independent X, Y so that $L(X, Y) = 0$ and $L(X, X) = L(Y, Y)$. We pick two such vectors X, Y in $Tp_m M$, apply ($*$) and obtain

$$\begin{aligned}\lambda^{-2}(1 - m^{-1})^2 &< \|L(X, X)\| \cdot \|L(Y, Y)\| \cdot \|X\|^{-2} \cdot \|Y\|^{-2} \\ &\leqslant (\langle L(X, X), L(Y, Y)\rangle - \|L(X, Y)\|^2) \cdot (\|X\|^2\|Y\|^2 - \langle X, Y\rangle^2)^{-1}.\end{aligned}$$

By the Gauss equation, the rightmost term in these inequalities is the sectional curvature of M at p_m for the plane spanned by X and Y. Now letting m go to infinity, we deduce $\lambda^{-2} \leqslant \limsup_m K(X \wedge Y)$ and thus prove the theorem.

It is noteworthy that the above proof includes a generalization of the following well-known result. If a compact hypersurface M in E^N is contained in a ball of radius λ, then there exists a point on M where all the normal curvatures are in absolute value not less than λ^{-1}. For a submanifold M of E^N, of arbitrary codimension, we define the absolute normal curvature at a point $p \in M$ and in the direction $X \in T_p M$, $\|X\| = 1$, to be $\|L(X, X)\|$. Let

$$C(p) = \min\{\|L(X, X)\|/X \in T_p M \text{ and } \|X\| = 1\}.$$

THEOREM 2. *Let M be a complete submanifold of E^N with sectional curvature bounded from below. If M is contained in a ball of radius λ, then* $\limsup_{p \in M} C(p) \geqslant \lambda^{-1}$.

PROOF. Apply Omori's theorem as in Theorem 1 to $\langle \varphi, \varphi\rangle/2$. From inequality ($*$) we immediately obtain the conclusion.

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