FUNDAMENTAL DIMENSION AND SUSPENSION OF APPROXIMATE FIBRATIONS

P. F. DUVAL¹ AND L. S. HUSCH

ABSTRACT. Given an approximate fibration $f: X \rightarrow Y$ between ANR's a new approximate fibration Sf is constructed with the property that the fibers of Sf are the suspensions of the fibers of f . This construction is used together with an approximation theorem of T. Chapman to give conditions under which $\text{Fd } X = \text{Fd } Y + \text{Fd } f^{-1}(x)$.

1. Introduction. The proper map $f: X \rightarrow Y$ between locally compact ANR's is an *approximate fibration* if whenever Z is a space and $\tilde{h}: Z \times \{0\} \rightarrow X$ and $H: Z \times I \rightarrow Y$ are maps such that $\tilde{f}\tilde{h} = H|_{Z \times \{0\}}$ and ϵ is an open cover of Y , there is an extension $\tilde{H}$ of $\tilde{h}$ to all of $Z \times I$ such that $f\tilde{H}$ and H are ϵ -close. If Y is path connected, the point inverses of F all have the same shape [CD1]. If F has the shape of $f^{-1}(x)$, we say that f is an approximate fibration *with fiber* F .

In this paper, we study the relation among the fundamental dimensions of F , X , and Y for an approximate fibration f with fiber F . In §2, we introduce a device for producing an approximate fibration $S^n f$ whose fiber is the n -fold suspension of F and use an approximation theorem of T. Chapman to show that if n is sufficiently large, the algebraic topology of $S^n f$ is essentially that of a locally trivial bundle. §3 is devoted to some algebraic preliminaries, and §4 is devoted to proving

THEOREM 6. *Suppose $f: X \rightarrow Y$ is an approximate fibration with fiber F , with X , Y continua and with $\tilde{\Pi}_1(F) \cong \tilde{\Pi}_1(Y) \cong 0$. Then $\text{Fd } X = \text{Fd } Y + \text{Fd } F$ provided that $\tilde{H}^k(F)$ is torsion free, where $k > 0$ is the fundamental dimension of F .*

We use $\tilde{\Pi}_i(X)$ to denote the i th shape group [B2] of a space X . Cohomology is with $\mathbb{Z}$ (= integer) coefficients if no coefficient group is indicated. $\text{Fd}(X)$ denotes the fundamental dimension of X , i.e., the least integer n such that X has the shape of an n -dimensional compactum. Our main tool for computing fundamental dimension is Nowak's theorem [N] that if X is an approximately 1-connected compactum (this is equivalent to $\tilde{\Pi}_1(X) = 0$ when X is a pointed FANR) and $\text{Fd } X < \infty$, then $\text{Fd } X = c(X)$, where $c(X) = \max\{n | \tilde{H}^n(X) \neq 0\}$. The symbol " $\cong$ " is used to denote isomorphism, homeomorphism, or shape equivalence, its exact meaning will either be clear from the context or be irrelevant.

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2. Suspending the fibers of an approximate fibration. Let $f: X \rightarrow Y$ be a proper surjective map between locally compact separable metric ANR's. Let Q be the Hilbert cube and $1: Q \rightarrow Q$ be the identity map. The following is easily proved [CD1].

PROPOSITION 1. *$f: X \rightarrow Y$ is an approximate fibration if and only if $f \times 1: X \times Q \rightarrow Y \times Q$ is an approximate fibration.*

Let SX be the space obtained by identifying two copies of the mapping cylinder $M(f)$ along their common top X . Formally, SX is the quotient space $(X \times [-1, 1] \amalg Y \times \{-1, 1\})/\sim$, where $\sim$ is the equivalence relation generated by $(x, i) \sim (f(x), i)$, $i = \pm 1$. Define $Sf: SX \rightarrow Y$ by $Sf([x, t]) = f(x)$. Note that for $y \in Y$, $Sf^{-1}(y)$ is homeomorphic with the suspension $\Sigma f^{-1}(y)$ of $f^{-1}(y)$. We define $S^n X$, $S^n f$ by iterating this construction. By [B1, p. 116], SX and, hence, $S^n X$ is a locally compact ANR for each n .

LEMMA 2. *Let $f: X \rightarrow Y$ be an approximate fibration between locally compact ANR's with connected fiber. Then Sf is an approximate fibration.*

PROOF. By Proposition 1, it suffices to show that $Sf \times 1: SX \times Q \rightarrow Y \times Q$ is an approximate fibration and by [CD2] it suffices to show that $Sf \times 1$ is k -movable for all k . By Edwards [E], $Y \times Q$ is a Q -manifold, hence, each point in $Y \times Q$ has a neighborhood base of contractible open sets.

We now claim that it suffices to show:

If $U \supseteq V$ are contractible open sets in $Y \times Q$,

$\Pi_i((Sf \times 1)^{-1}(U), (Sf \times 1)^{-1}(V)) \cong 0$ for each i and for

each base point in $(Sf \times 1)^{-1}(V)$. $[Sf \times 1: (SX) \times Q \rightarrow Y \times Q]$. (*)

PROOF OF CLAIM. Since each fiber of f is a pointed FANR [CD1], each fiber of Sf (and, hence, of $Sf \times 1$) is also a pointed FANR [DNS].

Let $b \in Y \times Q$ and let $\{U_i\}_{i=1}^\infty$ be a sequence of compact ANR neighborhoods of $F_b = (Sf \times 1)^{-1}(b)$ in $SX \times Q$ such that $U_{i+1} \subseteq \text{int } U_i$ for all i and $\bigcap_{i=1}^\infty U_i = F_b$. Fix $k > 0$. Since F_b is a pointed FANR, by choosing a subsequence, if necessary, we may assume that $e_{i*} | \text{image } e_{i+1*}: \text{image } e_{i+1*} \rightarrow \text{image } e_{i*}$ is an isomorphism for each i where e_{i*} is the homomorphism induced by inclusion $\Pi_j(U_i) \rightarrow \Pi_j(U_{i-1})$ for $1 \leq j \leq k+1$. In addition, the natural projection $\tilde{\Pi}_j(F_b) = \lim_{\leftarrow} \Pi_j(U_i) \rightarrow \Pi_j(U_i)$ induces an isomorphism $\tilde{\Pi}_j(F_b) \rightarrow \text{image } e_{i+1*}$ for all i and $1 \leq j \leq k+1$.

Let W be a contractible open set in $Y \times Q$ containing b such that for some i , $(Sf \times 1)^{-1}(W) \supseteq U_i$ and choose a contractible open neighborhood W_1 of b in $Y \times Q$ such that $U_i \supseteq (Sf \times 1)^{-1}(W_1)$. By using (*) and some algebra, it is easy to show that the composition $\tilde{\Pi}_j(F_b) \rightarrow \Pi_j(U_i) \rightarrow \Pi_j((Sf \times 1)^{-1}(W))$ is an isomorphism for $j \leq k$ and an epimorphism for $j = k+1$.

Recall that to show that $Sf \times 1$ is k -movable, we must show that for each $c \in Y \times Q$ and each neighborhood U_0 of the fiber F_c , there are neighborhoods U and V of F_c such that $V \subseteq U \subseteq U_0$ and $\tilde{\Pi}_1(F_b)$ is realized as the image of $\Pi_1(V)$ in

$\Pi_i(U)$, $0 \leq i \leq k$, for every fiber $F_b \subseteq V$. Given c and U_0 , choose a contractible open set W containing c in $Y \times Q$ such that $(Sf \times 1)^{-1}(W) \subseteq U_0$. Let $U = V = (Sf \times 1)^{-1}(W)$. For each b such that $F_b \subseteq V$, $\check{\Pi}_j(F_b) \rightarrow \Pi_j(V)$ is an isomorphism for $j \leq k$. Hence $Sf \times 1$ is k -movable and the claim is proved. Now we proceed to prove (*).

If $A \subseteq SX \times Q$, define $A_+ = \{([x, t], q) \in A \mid t \geq 0\}$, $A_- = \{([x, t], q) \in A \mid t \leq 0\}$, $A_0 = A_+ \cap A_-$. Now let $U \supseteq V$ be as in (*). If $y \in V$, the exact sequence from [CD1] gives the following commutative diagram where the vertical arrows are inclusion induced.

$$\begin{array}{ccccccc} \cdots \rightarrow & \check{\Pi}_i((f \times 1)^{-1}(y)) & \rightarrow & \Pi_i((f \times 1)^{-1}(V)) & \rightarrow & \Pi_i(V) & \rightarrow \cdots \\ & \downarrow & & \downarrow j_i & & \downarrow & \\ \cdots \rightarrow & \check{\Pi}_i((f \times 1)^{-1}(y)) & \rightarrow & \Pi_i((f \times 1)^{-1}(U)) & \rightarrow & \Pi_i(U) & \rightarrow \cdots \end{array}$$

Since U and V are contractible, it follows that j_i is an isomorphism for each i , so that $\Pi_*((f \times 1)^{-1}(U), (f \times 1)^{-1}(V)) \cong 0$. By the Hurewicz theorem,

$$H_*((f \times 1)^{-1}(U), (f \times 1)^{-1}(V)) \cong 0.$$

Now

$$H_*((f \times 1)^{-1}(U), (f \times 1)^{-1}(V)) \cong H_*((Sf \times 1)^{-1}(U)_0, (Sf \times 1)^{-1}(V)_0) \cong 0$$

and

$$H_*((Sf \times 1)^{-1}(U)_\pm, (Sf \times 1)^{-1}(V)_\pm) \cong H_*(U, V) \cong 0$$

so that $H_*((Sf \times 1)^{-1}(U), (Sf \times 1)^{-1}(V)) \cong 0$ by the Mayer-Vietoris sequence. Since $\Pi_1((Sf \times 1)^{-1}(U)) \cong 0$ by the Van Kampen theorem, we can apply the relative Hurewicz theorem to conclude that $\Pi_*((Sf \times 1)^{-1}(U), (Sf \times 1)^{-1}(V)) \cong 0$.

THEOREM 3. *Suppose that $p: X \rightarrow Y$ is an approximate fibration between locally compact ANR's with connected fiber F . Suppose that Y has the homotopy type of an n -complex, $n < \infty$. Then there is a locally trivial fibration $f: S^{n+1}X \times Q \rightarrow Y \times Q$ with fiber K such that*

(i) $\text{Sh } K = \text{Sh}(\Sigma^{n+1}F)$,

(ii) *there is a homotopy equivalence ϕ of $S^{n+1}X \times Q$ onto itself making*

$$\begin{array}{ccc} S^{n+1}X \times Q & \xrightarrow{\phi} & S^{n+1}X \times Q \\ S^{n+1}p \times 1 \searrow & & \swarrow f \\ & Y \times Q & \end{array}$$

commute.

PROOF. By Lemma 2, $S^{n+1}p \times 1$ is an approximate fibration with n -shape connected fibers. Thus by [C, Theorem 2], $S^{n+1}p \times 1$ is homotopic to (can be approximated by) a locally trivial map. Conclusions (i) and (ii) follow by applying the proof of Theorem 1.2 of [F] with $g = \text{id}$.

3. Algebraic lemmas.

LEMMA 4. *Let $f: X \rightarrow Y$ be an Hurewicz fibration with connected fiber F , where X and Y are compact connected ANR's and*

- (a) $\check{\Pi}_1(F) \cong \Pi_1(Y) \cong 0$,
- (b) $\text{Fd}(F) = k < \infty$, $\text{Fd}(Y) = l < \infty$, and
- (c) $H^k(F)$ is torsion free.

Then $\text{Fd}(X) = k + l$.

PROOF. It follows from the exact sequence of a fibration that $\Pi_1(X) \cong 0$, so by [N1], it suffices to show that $H^{k+l}(X) \neq 0$ and $H^n(X) \cong 0$ if $n > k + l$. The conclusion follows from a standard spectral sequence argument, which we sketch for completeness. We use the notation of [S]. In the cohomology spectral sequence of f , we have $E_2^{*,p,q} \cong H^p(Y; H^q(F))$, so that $E_r^{*,p,q} \cong E_\infty^{*,p,q} \cong 0$ if $p > l$ or $q > k$, and in particular, if $p + q = k + l$, $E_\infty^{*,p,q} \cong 0$ unless $p = l$ and $q = k$. Now $E_2^{*,l,k} \cong A = H^l(Y, H^k(F))$ and $A \neq 0$ by the universal coefficient theorem, since $H^l(Y) \neq 0$ and $H^k(F)$ is torsion free.

Suppose that we have shown that for some $r \geq 2$, $E_r^{*,l,k} \cong A$. Then $E_{r+1}^{*,l,k}$ is kernel mod image in

$$E_r^{*,l-r,k+r-1} \rightarrow A \rightarrow E_r^{*,l+r,k-r+1}.$$

Since the two extreme groups are zero, it follows that $E_{r+1}^{*,l,k} \cong E_\infty^{*,l,k} \cong A$.

Now $H^{k+l}(X)$ has a filtration

$$0 = D^{*,k+l+1,-1} \subset D^{*,k+l,0} \subset \dots \subset D^{*,k+l-i,i} \subset \dots \subset D^{*,0,k+l} \cong H^{k+l}(X),$$

where each $D^{*,k+l-i,i} / D^{*,k+l-i+1,i-1} \cong E_\infty^{*,k+l-i,i}$. It follows that $H^{k+l}(X) \cong A \neq 0$. A similar argument shows that $H^n(X) \cong 0$ if $n > k + l$.

REMARKS. If X and Y are merely assumed to be locally compact, then our argument shows that Lemma 4 remains true if "Fd" is replaced by "c" throughout the statement.

This lemma is surely known in its cohomological form by algebraic topologists; compare, for example [S, Proposition 3, p. 466] and [R, Lemma 6.1].

LEMMA 5. *Suppose $p: X \rightarrow Y$ is an approximate fibration between locally compact ANR's with connected fiber F , $c(F) = k > 0$. If $c(X) = m > c(Y)$, then (a) $c(\Sigma F) = k + 1$ and (b) $c(SX) = m + 1$.*

PROOF. (a) is immediate. For (b), note that the Mayor-Vietoris sequence of $(SX; S_+X, S_-X)$ gives an exact sequence

$$\dots \leftarrow H^{i+1}(Y) + H^{i+1}(Y) \leftarrow H^{i+1}(SX) \leftarrow H^i(X) \leftarrow H^i(Y) + H^i(Y) \leftarrow \dots$$

so that $H^{i+1}(SX) \cong H^i(X)$ if $i > c(Y)$.

4. Proof of Theorem 6.

THEOREM 6. *Suppose that $p: X \rightarrow Y$ is an approximate fibration with fiber F , where X and Y are compact connected ANR's with $\Pi_1(Y) \cong \check{\Pi}_1(F) \cong 0$, $\check{H}^k(F)$ torsion free, where $\text{Fd}(F) = k > 0$. Then $\text{Fd } X = \text{Fd } F + \text{Fd } Y$.*

PROOF. First, let us assume that F is connected. Let $\text{Fd } X = m$, $\text{Fd } Y = l$. Choose $N > \max(0, l - m)$ and let $\bar{X} = X \times T^N$ where T^N is the N -torus, the product of N copies of the 1-sphere, $\bar{p}: \bar{X} \rightarrow Y$ the composition $X \times T^N \xrightarrow{\text{proj}} X \xrightarrow{p} Y$. Then $\bar{p}$ is an approximate fibration with fiber $\bar{F} \cong F \times T^N$ [CD1]. By the Künneth formula $c(\bar{F}) = k + N$, $c(\bar{X}) = m + N$ and $\check{H}^{k+N}(\bar{F})$ is torsion free. (In fact, $\check{H}^{k+N}(F) \cong \check{H}^k(F)$.) Now choose n such that Y has the homotopy type of an $(n - 1)$ -complex. By Theorem 3, there is a Hurewicz fibration $f: S^n \bar{X} \times Q \rightarrow Y \times Q$ whose fiber has the shape of $\Sigma^n \bar{F}$. By Lemma 4,

$$\text{Fd}(S^n \bar{X} \times Q) = \text{Fd}(\Sigma^n \bar{F}) + \text{Fd}(Y),$$

hence, $c(S^n \bar{X} \times Q) = c(\Sigma^n \bar{F}) + c(Y)$. Lemma 5 gives that $c(\bar{X}) + n = c(\bar{F}) + n + c(Y)$, so that we have

$$c(X) + N + n = c(F) + N + n + c(Y), \quad \text{or} \quad c(X) = c(F) + c(Y);$$

$$\text{Fd } X = \text{Fd } F + \text{Fd } Y, \quad \text{by [N1].}$$

Now suppose that F is not connected. Let $p_0: X \rightarrow Y_0$ and $q: Y_0 \rightarrow Y$ be a monotone and a light mapping, respectively, such that $qp_0 = p$. It is easily shown that p_0 is an approximate fibration whose fiber is a component of F and q is a covering map. (It follows that the components of F have the same shape.) Apply the first part of this proof.

REMARKS. In Theorem 6 and Lemma 4 some homological assumption is necessary, for there are simple examples of 1-connected polyhedra X and Y with $\text{Fd}(X \times Y) < \text{Fd } X + \text{Fd } Y$ [N]. Our argument actually shows that Theorem 6 and Lemma 4 remain true if the condition $\check{H}^k(F)$ torsion free is replaced by $(*) H^l(Y; \check{H}^k(F)) \neq 0$. By tracing the role of $(*)$ through the several lemmas, one can also show that $(*)$ is necessary. As in the remarks after Lemma 4, Theorem 6 remains true for locally compact X and Y if "Fd" is replaced by "c".

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DEPARTMENT OF MATHEMATICS, OKLAHOMA STATE UNIVERSITY, STILLWATER, OKLAHOMA 74074

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF TENNESSEE, KNOXVILLE, TENNESSEE 37916