SELF-HOMOTOPY-EQUIVALENCES OF A SPACE WITH TWO NONVANISHING HOMOTOPY GROUPS

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ABSTRACT. The group $G_*(X)$ is isomorphic to $H^n(\pi_1(X); \pi_n(X))$ for the connected CW-complex X with two nonvanishing homotopy groups $\pi_1(X)$, $\pi_n(X)$, where $\pi_1(X)$ acts on $\pi_n(X)$ usually.

0. Introduction. Let X be a connected CW-complex with base point $*$. Then we have considered the group $G_*(X)$ of all based homotopy classes of self-homotopy-equivalences of X inducing the identity automorphisms of all homotopy groups (cf. [1], [11], [12]).

In this note, we show that $G_*(X) = H^j(K(\pi_i(X), i); \pi_j(X))$ for the connected CW-complex X with two nonvanishing homotopy groups $\pi_i(X)$, $\pi_j(X)$ ($j > i$), where $\pi_i(X)$ acts on $\pi_j(X)$ usually if $i = 1$ [Theorem 2.2].

This implies that to determine the group $G_*(X)$ is to compute the (local coefficient) cohomology of the group $\pi_i(X)$, that is, a problem in group theory.

For the simply connected case, the above result is shown by W. Shih [10] and later by Y. Nomura [8].

The method of the proof is different from that of the simply connected case.

Some examples in $G_*(X) = 1$ are discussed in [11] and other examples can be found in [3], [4] and [2].

In §1 we review J. F. McClendon [7] and the author [12], and in §2 we prove the above theorem; in §3 some examples of $G_*(X)$ are given.

Throughout this note, all spaces are base-pointed, and homotopies preserve base points, and we use the terminology of [12].

1. Twisted principal fibrations. Let $\phi: \pi_1(B) \rightarrow \text{Aut } G$ be the local coefficient system on B , and $p: \bar{P}_\theta \rightarrow B$ be the twisted principal fibration with fibre $K(G, n)$ (cf. [12, pp. 1–2]).

$$\begin{array}{ccccccc}
 \bar{P}_\theta & \rightarrow & \bar{P}L & & & & \\
 p \downarrow & & \downarrow & & & & \\
 B & \xrightarrow{\theta} & L_\phi(G, n+1) & \xrightarrow{q} & K = K(\pi_1(B), 1) & & (1.1) \\
 s \downarrow & & & & & & \\
 W & & & & & &
 \end{array}$$

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Associated with this diagram, we have the following.

PROPOSITION 1.2 ([7, p. 42]). *The following sequence is exact:*

$$[\bar{P}_\theta, \hat{\Omega}_W B]_B \xrightarrow{\delta} H^n(\bar{P}_\theta; G) \xrightarrow{\gamma} [\bar{P}_\theta, \bar{P}_\theta]_W \xrightarrow{p} [\bar{P}_\theta, B]_W.$$

In this sequence, $\hat{\Omega}_W B = F(S^1, B|s) = \{\omega: S^1 \rightarrow B | s\omega \text{ is constant}\}$, $[\ ,]_W$ and $[\ ,]_B$ denote the W -homotopy and B -homotopy sets, respectively, and $H^n(\bar{P}_\theta; G)$ is the cohomology with the local coefficient induced by

$$q\theta p: \bar{P}_\theta \rightarrow B \rightarrow L_\phi(G, n+1) \rightarrow K = K(\pi_1(B), 1).$$

There is an action $\nu_*: H^n(\bar{P}_\theta; G) \times [\bar{P}_\theta, \bar{P}_\theta]_W \rightarrow [\bar{P}_\theta, \bar{P}_\theta]_W$ (cf. [12, p. 3, (1.8)]), and we denote the isotropy group of the identity map $1_{\bar{P}_\theta}: \bar{P}_\theta \rightarrow \bar{P}_\theta$ by $I(1_{\bar{P}_\theta})$.

The map δ of Proposition 1.2 implies the following.

PROPOSITION 1.3 ([7, Theorem 3.4]). $\text{Im } \delta = I(1_{\bar{P}_\theta})$.

Put $W = K$ and $s = q\theta$ in the diagram (1.1). Then we have the following associated with the group $G_*(X)$.

PROPOSITION 1.4 ([12, Theorem 3.6]). *Assume that a connected CW-complex B satisfies $\pi_i(B) = 0$ for every $i > n$. Let $\phi: \pi_1(B) \rightarrow \text{Aut } G$ be the local coefficient system on B , and $p: \bar{P}_\theta \rightarrow B$ be the twisted principal fibration with fibre $K(G, n)$. Then, the following sequence of groups and homomorphisms is exact.*

$$1 \rightarrow I(1_{\bar{P}_\theta}) \xrightarrow{\subset} H^n(B; G) \xrightarrow{\Delta} G_*(\bar{P}_\theta) \xrightarrow{J} G_*(B),$$

where $H^n(B; G) = H^n(\bar{P}_\theta; G)$ is the cohomology with the local coefficient induced by

$$q\theta: B \rightarrow L_\phi(G, n+1) \rightarrow K = K(\pi_1(B), 1),$$

$\Delta(a) = \nu_*(a, 1_{\bar{P}_\theta})$ and J is a natural homomorphism (cf. [12, pp. 4–5]).

Considering a connected CW-complex X and its Postnikov-system $\{X_n\}$, let $\phi: \pi_1(X_{n-1}) = \pi_1(X) \rightarrow \text{Aut } \pi_n(X)$ be the usual local coefficient system on X_{n-1} , and put $B = X_{n-1}$, $W = K(\pi_1(X), 1) = K$ and $s = p_2 \circ \dots \circ p_{n-2} p_{n-1}: X_{n-1} \rightarrow X_1 = K(\pi_1(X), 1)$ in diagram (1.1). Associated with the fibration $p_n: X_n \rightarrow X_{n-1}$, let

$$[k] \in [X_{n-1}, L_\phi(\pi_n(X), n+1)]_K = H^{n+1}(X_{n-1}; \pi_n(X))$$

be the transgression image of the identity map $i_n \in [F, F]^\phi$, where $F = K(\pi_n(X), n)$ and $[F, F]^\phi$ denotes the equivariant homotopy classes under the action ϕ of $\pi_1(X)$ on $\pi_n(X)$ (cf. [6, p. 3]). We have a fibration $\bar{P}_k \rightarrow X_{n-1}$, then there exists a based homotopy equivalence $\eta: X_n \rightarrow \bar{P}_k$, which is a fibre map over X_{n-1} [12, Proposition 2.1]. And we have the following.

PROPOSITION 1.5 ([12, Corollary 3.7]). *Let X be a connected CW-complex and $\{X_n\}$ be a Postnikov-system of X . Then, the following sequence of groups and homomorphisms is exact for every $n > 1$.*

$$1 \rightarrow I(1_{X_n}) \xrightarrow{\subset} H^n(X_{n-1}; \pi_n(X)) \xrightarrow{\Delta} G_*(X_n) \xrightarrow{J} G_*(X_{n-1}),$$

where $I(1_{X_n}) = I(1_{\bar{F}_k})$, $H^n(X_{n-1}; \pi_n(X))$ is the cohomology with the local coefficients $\{\pi_n(X)\}$, where $\pi_1(X_{n-1}) = \pi_1(X)$ acts on $\pi_n(X)$ usually, and $\text{Im } J = \{[f_{n-1}] \in G_*(X_{n-1}) \mid [k] = f_{n-1}^*[k], [k] \in H^{n+1}(X_{n-1}; \pi_n(X))\}$.

2. Proof of the theorem. Let X be a connected CW-complex and $\{X_n\}$ be a Postnikov-system of X .

Let ϕ be the usual action of $\pi_1(X_1) = \pi_1(X)$ on $\pi_2(X)$, and put $B = X_1 = K(\pi_1(X), 1) = K$ in the diagram (1.1). Then we have the following diagram:

$$\begin{array}{ccccc} \bar{P}_\theta & \rightarrow & \bar{P}L & & \\ p\downarrow & & \downarrow & & \\ X_1 & \xrightarrow{\theta} & L_\phi(\pi_2(X), 3) & \xrightarrow{q} & K = K(\pi_1(X), 1) \\ \text{id}\downarrow & & & & \\ X_1 & & & & \end{array}$$

Let $\tau: [F, F]^\phi \rightarrow [X_1, L_\phi(\pi_2(X), 3)]_K = H^3(X_1; \pi_2(X))$ be the transgression. Then we have the following.

THEOREM 2.1. *If $\theta \in \text{Im } \tau$, $I(1_{\bar{P}_\theta}) = 1$. Therefore, we have the following exact sequence for $\theta \in \text{Im } \tau$.*

$$1 = I(1_{\bar{P}_\theta}) \xrightarrow{\subseteq} H^2(X_1; \pi_2(X)) \xrightarrow{\Delta} G_*(\bar{P}_\theta) \xrightarrow{J} G_*(X_1) = 1,$$

where $H^2(X_1; \pi_2(X))$ is the cohomology with the local coefficients $\{\pi_2(X)\}$, where $\pi_1(X_1) = \pi_1(X)$ acts on $\pi_2(X)$ usually.

PROOF. The domain of the map δ is $[\bar{P}_\theta, \hat{\Omega}_{X_1} X_1]_{X_1}$. Since $s = \text{id}$, $\hat{\Omega}_{X_1} X_1$ consists of the constant path at every point of X_1 .

Consider the following diagram:

$$\begin{array}{ccc} \bar{P}_\theta & \rightarrow & \hat{\Omega}_{X_1} X_1 \\ p \searrow & & \swarrow \\ & X_1 & \end{array}$$

We obtain that $[\bar{P}_\theta, \hat{\Omega}_{X_1} X_1]_{X_1}$ consists of the single element, that is, $[\bar{P}_\theta, \hat{\Omega}_{X_1} X_1]_{X_1} = *$, where $\hat{\Omega}_{X_1} X_1 \rightarrow X_1$ is $\omega \rightarrow \omega(p)$, $p = (1, 0) \in S^1$.

Hence, $\text{Im } \delta = I(1_{\bar{P}_\theta}) = 1$. Since $X_1 = K(\pi_1(X), 1)$, $G_*(X_1) = 1$. Therefore we have the desired sequence. Q.E.D.

Since the identity map $i_2 \in [F, F]^\phi$, $I(1_{X_2}) = 1$. Therefore, we have the following theorem by Proposition 1.5 and the above theorem.

THEOREM 2.2. $G_*(X) = H^j(K(\pi_i(X), i); \pi_j(X))$ for the connected CW-complex X with two nonvanishing homotopy groups $\pi_i(X)$, $\pi_j(X)$ ($j > i$), where $\pi_i(X)$ acts on $\pi_j(X)$ usually if $i = 1$.

Furthermore we have the following exact sequence, which is shown by B. Schellenberg [9], A. J. Sieradski and M. N. Dyer [3], [4] for the n -dimensional compact connected CW-complex X such that $\pi_i(X) = 0$ for $1 < i < n$. Incidentally let $G(X)$ be the group of all based homotopy classes of self-homotopy-equivalences

of the n -dimensional connected CW-complex X . Then $G(X_n) = G(X)$. And X_n has two nonvanishing homotopy groups, trivially.

Considering the following exact sequence

$$1 \rightarrow G_*(X_n) \rightarrow G(X_n) \rightarrow \text{Aut } \pi_1(X) \times \text{Aut } \pi_n(X),$$

the above theorem implies the following.

THEOREM 2.3. *If X is an n -dimensional CW-complex such that $\pi_i(X) = 0$ for $1 < i < n$, the following sequence is exact:*

$$1 \rightarrow H^n(\pi_1(X); \pi_n(X)) \rightarrow G(X) \rightarrow \text{Aut } \pi_1(X) \times \text{Aut } \pi_n(X),$$

where $\pi_1(X)$ acts on $\pi_n(X)$ in the usual manner.

Moreover, we have the following theorem by Proposition 1.5.

THEOREM 2.4. *The following sequence is exact.*

$$1 \rightarrow I(1_{X_3}) \xrightarrow{\subset} H^3(X_2; \pi_3(X)) \xrightarrow{\Delta} G_*(X_3) \xrightarrow{J} H^2(\pi_1(X); \pi_2(X))$$

and $\text{Im } J = \{[f_2] \in G_*(X_2) = H^2(\pi_1(X); \pi_2(X)) \mid [k] = f_2^*[k], [k] \in H^4(X_2; \pi_3(X))\}$.

3. Some examples of $G_*(X)$.

EXAMPLE 3.1 ([2, p. 79]). If X is a connected CW-complex with two nonvanishing homotopy groups $\pi_1(X) = G$ (finite cyclic group), $\pi_n(X) = M$ ($n > 1$),

$$G_*(X) = \begin{cases} M^G/TM, & n \text{ even,} \\ M_T/IM, & n \text{ odd,} \end{cases}$$

where M^G is the submodule of elements fixed under the action of G , TM is the image of the trace map $T: M \rightarrow M^G$, $M_T = \{m \in M \mid Tm = 0\}$, and, IM is the submodule generated by $\{g'm \mid g' \in I, m \in M\}$, where I is the augmentation ideal in the group ring ZG .

EXAMPLE 3.2. Every n -dimensional cohomology $H^n(X; G_x)$ with the local coefficients $\{G_x\}$ of a connected CW-complex X is isomorphic to $G_*(V_{G_x}^n)$ of a suitable CW-complex $V_{G_x}^n$, where $V_{G_x}^n$ depends on the dimension n and the local coefficients $\{G_x\}$.

PROOF. By [5, Theorem 1.1], we have the following fact, that is, for every path connected space X with base point $*$, there exists a (Serre) fibration $TX \xrightarrow{\iota_X} X$, and the map ι_X induces an isomorphism on cohomology with local coefficients, i.e., $H^n(TW; G_x) = H^n(X; G_x)$ for every $n > 1$, and $\pi_i(TX) = 0$ for $i > 1$.

We can construct a CW-complex $V_{G_x}^n$ such that

$$\begin{aligned} \pi_1(V_{G_x}^n) &= \pi_1(TX), \\ \pi_n(V_{G_x}^n) &= G_*, \\ \pi_i(V_{G_x}^n) &= 0 \quad \text{otherwise,} \end{aligned}$$

and π_1 action on G_* is induced by $\pi_1(TX) \xrightarrow{\iota_X^*} \pi_1(X)$.

Trivially, the result is obtained by applying Theorem 2.2. Q.E.D.

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