NONCOMPACT HEREDITARILY STRONGLY INFINITE DIMENSIONAL SPACES

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ABSTRACT. It is shown that any strongly infinite dimensional space contains a strongly infinite dimensional subspace all of whose subspaces are either 0-dimensional or strongly infinite dimensional.

1. Introduction. In [R] we proved the existence of a compact strongly infinite dimensional subspace of the Hilbert cube each of whose subspaces is either 0-dimensional or strongly infinite dimensional. This result, perhaps unexpected, was a continuation of a pattern of results found in [Bi], [He], [Z], [R-S-W], [W]. To some extent, upon carefully examining the proofs in these references, one might become convinced that compactness is a crucial element. That this is not the case and that the existence of such examples is truly dimension-theoretic, will be shown herein. We shall demonstrate that each strongly infinite dimensional space (compact or not) contains a strongly infinite dimensional closed subspace that is hereditarily strongly infinite dimensional in the sense that each of its subspaces is either 0-dimensional or strongly infinite dimensional.

In §4 of [A], Aleksandrov asked whether every infinite dimensional metric compactum is either countable dimensional or strongly infinite dimensional. Although this question has not been answered, Roman Pol, assuming the continuum hypothesis, recently proved the existence of an infinite dimensional separable metric space, which is neither countable dimensional nor strongly infinite dimensional. His argument, which is not published, uses elements of the proof in [H]. It is interesting that a hereditarily strongly infinite dimensional space cannot contain such spaces.

2. Preliminaries. The spaces dealt with here are all separable and metrizable. Let Γ denote a countable or finite indexing set.

2.1. DEFINITION. Let X be a space and A, B be closed subsets of X . A closed set S of X *separates* A and B if $X - S$ can be written as the union of two separated sets, one containing A , the other containing B . We say S *continuum-wise separates* A and B if each continuum in X that meets A and B also meets S .

2.2. DEFINITION. Let X be a space and $\{(A_k, B_k) | k \in \Gamma\}$ be a collection of disjoint pairs of closed, nonempty subsets of X . The family is called *essential* provided that if S_k separates A_k and B_k for each k , then $\bigcap \{S_k | k \in \Gamma\} \neq \emptyset$. If Γ is

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infinite, then X is called *strongly infinite dimensional* if it has such an essential family.

2.3. For each $n \geq 1$, let $I_n = [-1, 1]$; then $Q = \prod \{I_n | n \geq 1\}$ is the Hilbert cube and $\pi_n: Q \rightarrow I_n$ is the n th coordinate projection. The sets $A_n = \pi_n^{-1}(-1)$ and $B_n = \pi_n^{-1}(1)$ are called the *opposite faces* of Q in the I_n direction. Using the proof on p. 49 of [H-W], one sees that $\{(A_n, B_n) | n \geq 1\}$ is an essential family in Q . Hence the Hilbert cube is strongly infinite dimensional. Let

$$A_n^+ = \pi_n^{-1}\left[-1, \frac{1}{2}\right], \quad B_n^+ = \pi_n^{-1}\left[\frac{1}{2}, 1\right].$$

3. Main result. We are now ready to state and prove our main result.

3.1. THEOREM. *Let T be a strongly infinite dimensional space. Then T contains a closed subspace which is hereditarily strongly infinite dimensional.*

PROOF. Let $\{(\mathcal{Q}_k, \mathcal{B}_k) | k \geq 1\}$ be an essential family for T . Embed T in Q so that for each k there is $m(k)$ such that $\mathcal{Q}_k \subset A_{m(k)}$, $\mathcal{B}_k \subset B_{m(k)}$. For example, one may use the Embedding Lemma, 3.5 of [K].

Using the techniques from [R], for each $k \geq 2$, there is a closed set Z_k of Q satisfying

1. Z_k continuum-wise separates $A_{m(k)}^+$ and $B_{m(k)}^+$;
2. $Z_k \cap A_{m(k)} = \emptyset$ and $Z_k \cap B_{m(k)} = \emptyset$;
3. $Z = \bigcap \{Z_k : k \geq 2\}$ contains only 0-dimensional and strongly infinite dimensional subsets.

The desired example is $X = T \cap Z$. Note that Z_k separates $A_{m(k)}$ and $B_{m(k)}$ and hence $Z_k \cap T$ separates $\mathcal{Q}_k$ and $\mathcal{B}_k$ in T ; in particular, $\dim X \geq 1$ since if $\dim X < 0$, then there is a closed subset Z_1 of T separating $\mathcal{Q}_1$ and $\mathcal{B}_1$ with $Z_1 \cap X = \emptyset$ and, therefore, $Z_1 \cap (\bigcap \{Z_k \cap T : k \geq 2\}) = \emptyset$.

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