ON AN ANALOG OF LAGRANGE'S THEOREM FOR COMMUTATIVE HOPF ALGEBRAS

DAVID E. RADFORD¹

ABSTRACT. We show that the Hopf algebra A over any field k representing the affine group scheme $SL(2, \quad)$ is not a free B -module for some sub-Hopf algebra B of A . In particular k can be algebraically closed, or of characteristic 0 in which case A is also cosemisimple.

Introduction. Let G be a finite group and suppose that H is a subgroup of G . If k is a field, then Lagrange's theorem can be formulated in terms of the group algebra $k[G]$, namely $k[G]$ is a free (left) $k[H]$ -module. An obvious Hopf algebra analog would be the following: If A is a Hopf algebra over a field k and B is a sub-Hopf algebra of A , then A is a free (left) B -module. This is not generally true; there exist commutative cocommutative counterexamples over certain nonalgebraically closed fields [1]. By the theorem of [3], the field of definition of a cocommutative counterexample can never be algebraically closed.

In this paper we show that the Hopf algebra A over a field k representing the affine group scheme $SL(2, \quad)$ is a finitely generated B -module which is not free for some sub-Hopf algebra B of A . In particular, k can be algebraically closed. In characteristic 0 observe that A is cosemisimple [7]. Our proof is very elementary and reduces to simple computations in the polynomial ring $k[X]$.

It should be noted that a commutative Hopf algebra A over a field k is always a faithfully flat and projective B -module for any sub-Hopf algebra B [5, Theorem 3.1] and [6, Corollary 1], and if B is finite-dimensional then A is a free B -module [3, Theorem 1]. For further results on the analog of Lagrange's theorem the reader is referred to [2], [3], [5], [6].

The main result. Let k be a field and $n \geq 1$ a fixed integer. Let C be the vector space with basis of symbols X_{ij} , $1 \leq i, j \leq n$. Then C is a coalgebra where $\Delta X_{ij} = \sum_k X_{ik} \otimes X_{kj}$ and $\epsilon(X_{ij}) = \delta_{ij}$ for all i, j . The coalgebra structure on C extends uniquely to a bialgebra structure on the symmetric algebra $S(C)$ of C . It is well known that the determinant form $d = \sum_{\sigma} \text{sgn}(\sigma) X_{1\sigma(1)} \cdots X_{n\sigma(n)}$ satisfies $\Delta d = d \otimes d$, and the quotient $A = S(C)/(d - 1)$ has a Hopf algebra structure. It is easily seen that A represents $SL(n, \quad)$. For the remainder of this paper we let $n = 2$.

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THEOREM. *Let k be any field and $A = (C)$ be as above. If $B = (C^2)$ is the subalgebra generated by C^2 , then B is a sub-Hopf algebra, and A is a finitely generated B -module which is not free.*

PROOF. C^n is a subcoalgebra for $n \geq 1$, so $B = k + C^2 + C^4 + \dots$ is a sub-bialgebra. If s is the antipode of A then $s(C) \subset C$, so B is in fact a sub-Hopf algebra. It is clear that $k + C$ generates $A = k + C + C^2 + C^3 + \dots$ as a B -module. It remains to show that A is not free. Let $M = C + C^3 + C^5 + \dots$. Then M is a B -module and:

(1) $A = B \oplus M$.

To prove (1) regard $S(C) = k[X_{11}, X_{12}, X_{21}, X_{22}]$ as a polynomial ring. Let $S(C)_0$ be the span of monomials of even degree and $S(C)_1$ be the span of monomials of odd degree (X_{ij} has degree 1). Then $S(C) = S(C)_0 \oplus S(C)_1$, and if $S(C) \xrightarrow{\pi} A$ is the projection, we have $\pi(S(C)_0) = B$ and $\pi(S(C)_1) = M$. If $I \subset S(C)$ is the ideal generated by $d - 1 = X_{11}X_{22} - X_{12}X_{21} - 1$, then $I = I_0 \oplus I_1$ where $I_i = I \cap S(C)_i$ for $i = 0, 1$. This proves (1).

(2) *If A is a free B -module then M is a free rank 1 B -module.*

To prove this let $J = \ker \epsilon_B$ be the augmentation ideal of B . Assume A is a free B -module. Then A/JA is a free B/J -module of the same rank. By (1) $A/JA \simeq B/J \oplus M/JM$, so the rank of A/JA as a $B/J \simeq k$ -module is $1 + \dim M/JM$.

We first show that A has rank 2 by showing $\dim M/JM = 1$. Since $\epsilon(x_{11}) = \epsilon(x_{22}) = 1$ and $\epsilon(x_{12}) = \epsilon(x_{21}) = 0$ (small case letters denote cosets of $A = S(C)/I$), and M is spanned by cosets represented by monomials of odd degree, M/JM is spanned by $\bar{x}_{11}$, $\bar{x}_{12}$, $\bar{x}_{21}$, and $\bar{x}_{22}$. Now multiplying the equation $x_{11}x_{22} - x_{12}x_{21} = 1$ by x_{11} , x_{12} , and x_{21} respectively, we deduce that $x_{11} - x_{22}$, x_{12} , $x_{21} \in JM$. Therefore $\dim M/JM = 1$.

Let $u_1, u_2 \in A$ be a basis for A over B , and write $u_i = b_i \oplus m_i$ for $i = 1, 2$. For some $c_1, c_2 \in B$ we have $1 = c_1u_1 + c_2u_2$, so $1 = c_1b_1 + c_2b_2$ and $c_1m_1 + c_2m_2 = 0$. Thus

$$\begin{bmatrix} c_1 & c_2 \\ -b_2 & b_1 \end{bmatrix} \in M(2, B)$$

is invertible, and if $m = -b_2m_1 + b_1m_2 \in M$, the computation

$$\begin{bmatrix} c_1 & c_2 \\ -b_2 & b_1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 1 \\ m \end{bmatrix}$$

shows that $1, m$ also form a basis for A . This concludes the proof of (2).

We will show that M is not a cyclic B -module; thus the theorem will be proved by (2). Let $\underline{A} = k[X, Y, Z]/(Z^2 - XY - 1)$ and $V \subset \underline{A}$ be the span of x, y, z . Let $\underline{B} = (V^2)$ and $\underline{M} = V + V^3 + V^5 + \dots$, and let $\underline{A} \xrightarrow{\pi} \underline{A}$ be the algebra homomorphism determined by $x_{11}, x_{22} \mapsto z$, $x_{12} \mapsto x$, and $x_{21} \mapsto y$. It is clear that $\pi(\underline{B}) = \underline{B}$ and $\pi(\underline{M}) = \underline{M}$. To prove that M is not a cyclic B -module we need only to show that:

(3) $\underline{M}$ is not a cyclic $\underline{B}$ -module.

First note that $\underline{A}$ is an extension of $k[X, Y]$, and is in fact a free $k[X, Y]$ -module with basis $1, z$. Assume that $\underline{M}$ is cyclic with generator $\underline{m}$. Then $\underline{m} = p + qz$ where

$p = p(X, Y)$, $q = q(X, Y) \in k[X, Y]$, and p is a linear combination of monomials of odd degree. In particular, $p \in k$ implies $p = 0$. Now let $\underline{b} = p' + q'z \in \underline{B}$. Then $\underline{bm} = (pp' + qq'(XY + 1)) + (pq' + p'q)z$, so since $X, z \in \underline{M}$, the equations

- (a) $pp' + qq'(XY + 1) = X$,
- (b) $pq' + p'q = 0$,
- (a') $pp'' + qq''(XY + 1) = 0$,
- (b') $pq'' + p''q = 1$,

must have solutions for some $p', p'', q', q'' \in k[X, Y]$. To show there are no solutions to the equations we may assume k is algebraically closed.

First note that since $p \in k$ implies $p = 0$, by (b') we deduce that $q \neq 0$. Let $\alpha \in k \setminus 0$ be such that $q(X, \alpha) \neq 0$. By virtue of (a') and (b') we have $p|q''(XY + 1)$ and $q|p''$. By (a) $\alpha X + 1$ does not divide $p(X, \alpha)$, so we conclude that $p(X, \alpha)|q''(X, \alpha)$. Write $p''(X, \alpha) = q(X, \alpha)u$ and $q''(X, \alpha) = p(X, \alpha)v$ where $u, v \in k[X]$. Then by (a') we calculate $p(X, \alpha)q(X, \alpha)(u + v(\alpha X + 1)) = 0$. Since $p(X, \alpha) \neq 0$ by (a) we have $u + v(\alpha X + 1) = 0$. Therefore by (b') we have $p(X, \alpha)^2v - q(X, \alpha)^2v(\alpha X + 1) = 1$, or

$$(4) \quad p(X, \alpha)^2 - q(X, \alpha)^2(\alpha X + 1) \in k.$$

But this last equation is not possible since $p(x, \alpha)^2$ has even degree and $q(X, \alpha)^2(\alpha X + 1)$ has odd degree. This proves (3), and the proof of the theorem is complete.

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ILLINOIS, CHICAGO CIRCLE, ILLINOIS 60680

Current address: Department of Mathematics, Rutgers University, New Brunswick, New Jersey 08903