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ZEROS OF SUCCESSIVE DERIVATIVES AND ITERATED OPERATORS ON ANALYTIC FUNCTIONS

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ABSTRACT. For a function f analytic in the closed disc $|z| \leqslant 1$, we study the behavior of zeros of the successive iterates $(\theta^n f)(z)$, $n = 0, 1, 2, \ldots$, where $\theta = (z + \alpha)^{p+1} d/dz$. We find that such behavior closely parallels that for the ordinary derivative operator. Using change-of-variable methods, we obtain information on zeros of derivatives of functions analytic in half-planes.

1. Introduction. Let f be analytic in the unit disc $|z| < 1$. A well-known principle in function theory is that f cannot have too many derivatives vanishing too near $z = 0$, unless f is a polynomial. The study of this phenomenon is the theory which has been associated with the names of Gončarov and Whittaker [2]–[5]. Its principal feature is the existence of zero-free neighborhoods of $z = 0$. There is an absolute constant G, known as the *Gončarov constant*, such that if f is analytic in $|z| < 1$, is not a polynomial, and $\varepsilon > 0$, then there is an infinite sequence of derivatives $f^{(n_k)}(z)$ which do not vanish in the discs $|z| < (G - \varepsilon)/(n_k + 1)$. The exact value of G is unknown, but it is known to lie between .7259 and .7378.

In the present paper, we consider the analogous problem for the case of the differential operator $\theta = (z + \alpha)^{p+1} d/dz$, where $p \geqslant 0$ and where α denotes a complex number. Taking $|\alpha| \leqslant 1$ and $f(z)$ analytic in a neighborhood of the closed unit disc $|z| \leqslant 1$, we study the zero-free regions of the iterates $(\theta^n f)(z)$, $n = 0, 1, 2, \ldots$. The neighborhoods of $z = -\alpha$ are the most interesting, for in this case all but a finite number (not just an infinite number) of the iterates $(\theta^n f)(z)$ are nonzero in punctured discs which shrink with increasing n to the point $z = -\alpha$.

The results we obtain for differentiation do not arise from taking $p = -1$ in the definition of θ. Instead, we use other values of p and employ change-of-variable methods to get information about zeros of derivatives of functions analytic in regions of the plane other than discs. Such problems have been studied by Widder [7, Theorem 31 and corollary, pp. 166–167] for functions analytic at ∞, and for functions analytic in half-planes and representable as Laplace Transforms, and a simpler proof of a result implied by Widder has recently been given by Boas [1]. This result can be stated as follows.

THEOREM A ([1], [7]). *Let* $F(w) = \sum_{n=1}^{\infty} b_n w^{-n}$ *be analytic at* ∞, *with* F *nonconstant. Then there is a constant* $c > 0$ *such that for all* n *sufficiently large,* $F^{(n)}(w)$ *has no finite zero outside the circle* $|w| = nc$.

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As F is analytic in $|z| > R$ in Theorem A, the constant c depends on F and R.

Our results yield this theorem as a special case. We also obtain Widder's description of the radial distribution of zeros [7, Theorem 35], and we get additional theorems on periodic functions, where Widder's method does not apply. These applications are given in §3.

2. The θ-operator. We shall obtain an integral representation for the nth iterate $(\theta^n f)(z)$. There are various ways of getting to the end result; for example, one could extend the method used by Hille [6, Vol. 2, p. 51] for the operator $z\,d/dz$. We will use an alternate approach.

We do this by expressing $\theta^n[z^k]$ as a function of $(z+\alpha)^{1+np}$ having polynomial coefficients, as given in (2.2), and then express the polynomials as integral transforms. Then the integral representation for $\theta^n f$ is obtained by the usual power series method.

For each nonnegative integer m, we have $\theta[(z+\alpha)^m] = m(z+\alpha)^{m+p}$, and more generally,

$$\begin{aligned}\theta^n\left[(z+\alpha)^m\right] &= (m)(m+p)(m+2p)\cdots(m+(n-1)p)(z+\alpha)^{m+np}\\ &= C_{mn}^{(p)}(z+\alpha)^{m+np}, \qquad m \geqslant 0, n \geqslant 1, \qquad (2.1)\end{aligned}$$

where $C_{mn}^{(p)} = (m)(m+p)(m+2p)\cdots(m+(n-1)p)$. If we apply (2.1) to the binomial expansion

$$z^k = \sum_{m=0}^{k} (-1)^m \binom{k}{m} (z+\alpha)^{k-m}\alpha^m,$$

and then put $z+\alpha = -\zeta$, we get

$$\begin{aligned}\theta^n(z^k) &= \sum_{m=0}^{k-1} (-1)^m \binom{k}{m} C_{k-m,n}^{(p)} (z+\alpha)^{k-m+np}\alpha^m\\ &= (z+\alpha)^{1+np} \sum_{m=0}^{k-1} (-1)^m \binom{k}{m} \alpha^m C_{k-m,n}^{(p)} (-1)^{k-m-1}\zeta^{k-m-1}\\ &= (z+\alpha)^{1+np}(-1)^{k-1}\left\{\sum_{m=0}^{k-1} \binom{k}{m}\alpha^m C_{k-m,n}^{(p)}\zeta^{k-m-1}\right\}\\ &= (z+\alpha)^{1+np}(-1)^{k-1}P_{k-1}^{(n)}(\zeta), \qquad n \geqslant 1, k \geqslant 1, \qquad (2.2)\end{aligned}$$

with $P_{k-1}^{(n)}(\zeta)$ defined in the indicated way. Of course, $P_{k-1}^{(n)}(\zeta)$ also depends on α and p, but we suppress this dependence to simplify notation. From the definition of $C_{mn}^{(p)}$ we see that

$$0 \leqslant C_{mn}^{(p)} \leqslant m^n[1+(n-1)p]^n, \qquad (2.3)$$

and so

$$\begin{aligned}|P_{k-1}^{(n)}(\zeta)| &\leqslant [1+(n-1)p]^n \sum_{m=0}^{k} \binom{k}{m}|\alpha|^m(k-m)^n|\zeta|^{k-m-1}\\ &\leqslant |\zeta|^{-1}[1+(n-1)p]^n k^n(|\alpha|+|\zeta|)^k, \qquad n \geqslant 1, k \geqslant 1.\end{aligned}$$

For each fixed ζ, then, the power series

$$G_n(\zeta, t) = \sum_{k=1}^{\infty} P_{k-1}^{(n)}(\zeta) t^{k-1} \tag{2.4}$$

(which also depends on α and p) converges at least in the disc $|t| < (|\alpha| + |\zeta|)^{-1}$. Now substitute the defining expression for $P_{k-1}^{(n)}(\zeta)$ into (2.4) and formally interchange the order of summation. This leads to

$$\begin{aligned}
\sum_{k=1}^{\infty} P_{k-1}^{(n)}(\zeta) t^{k-1} &= \sum_{k=1}^{\infty} \left\{ \sum_{r=0}^{k-1} \binom{k}{r} \alpha^r C_{k-r,n}^{(p)} \zeta^{k-r-1} \right\} t^{k-1} \\
&= \sum_{k=1}^{\infty} t^{k-1} \sum_{m=1}^{k} \binom{k}{k-m} \alpha^{k-m} C_{mn}^{(p)} \zeta^{m-1} \\
&= \sum_{m=1}^{\infty} C_{mn}^{(p)} \zeta^{m-1} t^{m-1} \sum_{k=m}^{\infty} \binom{k}{k-m} \alpha^{k-m} t^{k-m} \\
&= \sum_{m=1}^{\infty} C_{mn}^{(p)} (\zeta t)^{m-1} \sum_{r=0}^{\infty} \binom{m+r}{r} (\alpha t)^r \\
&= \sum_{m=1}^{\infty} C_{mn}^{(p)} (\zeta t)^{m-1} (1 - \alpha t)^{-(m+1)} \\
&= (1 - \alpha t)^{-2} \sum_{m=1}^{\infty} C_{mn}^{(p)} [(\zeta t)/(1 - \alpha t)]^{m-1}.
\end{aligned} \tag{2.5}$$

In view of (2.3) it follows that the interchange in order of summation will be justified when $|\alpha t| + |\zeta t| < 1$. Note that $|\alpha t| + |\zeta t| < 1$ implies $|\zeta t| < 1 - |\alpha t|$, which implies

$$\left| \frac{\zeta t}{1 - \alpha t} \right| < \frac{1 - |\alpha t|}{|1 - \alpha t|} \leqslant 1.$$

In particular, (2.4) and (2.5) are both valid, and we have

$$G_n(\zeta, t) = \sum_{k=1}^{\infty} P_{k-1}^{(n)}(\zeta) t^{k-1} = (1 - \alpha t)^{-2} \sum_{m=1}^{\infty} C_{mn}^{(p)} \left[\frac{\zeta t}{1 - \alpha t} \right]^{m-1},$$
$$\text{for } |t| < 1/(|\alpha| + |\zeta|). \tag{2.6}$$

Let ζ be fixed and let the real number r satisfy $0 < r < (|\alpha| + |\zeta|)^{-1}$. Then the Cauchy Integral Formula applied to (2.6) gives

$$P_{k-1}^{(n)}(\zeta) = \frac{1}{2\pi i} \int_{|t|=r} \frac{G_n(\zeta, t)}{t^k} \, dt, \qquad n \geqslant 1, k \geqslant 1. \tag{2.7}$$

THEOREM 2.1. *Let $f(z) = \sum_{k=0}^{\infty} a_k z^k$ be analytic in the disc $|z| < R$, where $R > 1$, let r satisfy $1 < r^{-1} < R$, and let ζ and α satisfy $|\zeta| + |\alpha| < r^{-1}$. If $z + \alpha = -\zeta$, then*

$$(\theta^n f)(z) = -\frac{(z + \alpha)^{1+np}}{2\pi i} \int_{|t|=r} f\left(-\frac{1}{t}\right) G_n(\zeta, t) \, dt, \qquad n = 1, 2, 3, \ldots. \tag{2.8}$$

PROOF. Using (2.2) and (2.7), apply θ^n termwise to the power series for $f(z)$ to obtain

$$(\theta^n f)(z) = \sum_{k=1}^{\infty} a_k \theta^n(z^k) = (z + \alpha)^{1+np} \sum_{k=1}^{\infty} (-1)^{k-1} a_k P_{k-1}^{(n)}(\zeta)$$

$$= (z + \alpha)^{1+np} \sum_{k=1}^{\infty} \frac{(-1)^{k-1} a_k}{2\pi i} \int_{|t|=r} \frac{G_n(\zeta, t)}{t^k}\, dt.$$

Since $|t^{-1}| = r^{-1} < R$ in the range of integration, uniform convergence gives

$$(\theta^n f)(z) = \frac{(z+\alpha)^{1+np}}{2\pi i} \int_{|t|=r} \left\{ \sum_{k=1}^{\infty} \frac{(-1)^{k-1} a_k}{t^k} \right\} G_n(\zeta, t)\, dt$$

$$= \frac{(z+\alpha)^{1+np}}{2\pi i} \int_{|t|=r} \left\{ f(0) - f\left(-\frac{1}{t}\right) \right\} G_n(\zeta, t)\, dt.$$

Since $G_n(\zeta, t)$ is analytic for $|t| < (|\alpha| + |\zeta|)^{-1}$, the term involving $f(0)$ drops out, giving (2.8).

The representation (2.8) extends the analogous formula of Hille [6] mentioned earlier in connection with the operator $z\, d/dz$. The terms in (2.8) are also defined for $p = -1$, and when $\alpha = 0$ the equation reduces to the Cauchy Integral Formula for derivatives. However, all the results given below require $p > 0$, and so we retain this assumption throughout.

We are now going to replace ζ in (2.8) by an indexed variable ζ_n so as to make the sequence $G_n(\zeta_n, t)$ converge, as $n \to \infty$. The choice of ζ_n is suggested by the following lemma.

LEMMA 2.1. *For fixed $m > 1$, and $p > 0$, the sequence*

$$S_{mn}^{(p)} = \frac{C_{mn}^{(p)}}{C_{1n}^{(p)}} \left[\frac{C_{1n}^{(p)}}{C_{2n}^{(p)}} \right]^{m-1} \qquad (n = 1, 2, 3, \dots)$$

is convergent. Moreover $(C_{1n}^{(p)}/C_{2n}^{(p)}) \to 0$, $n \to \infty$.

PROOF. If $n = 1$, we have

$$S_{m1}^{(p)} = (m/2^{m-1}) < 1, \qquad m > 1.$$

Next, observe that $S_{m,n+1}^{(p)}$ is obtained from $S_{mn}^{(p)}$ by multiplying by the factor

$$\frac{m + np}{1 + np} \left[\frac{1 + np}{2 + np} \right]^{m-1}.$$

We claim that this factor is at most 1, for that would be equivalent to

$$1 + \frac{m-1}{1+np} < \left[1 + \frac{1}{1+np} \right]^{m-1},$$

which is true owing to the binomial theorem. So the terms $S_{mn}^{(p)}$ satisfy $0 < S_{mn}^{(p)} < 1$ and are monotone decreasing as $n \to \infty$. The first conclusion follows. As for the second, note that

$$\frac{C_{2n}^{(p)}}{C_{1n}^{(p)}} = \prod_{k=1}^{n} \left[1 + \frac{1}{1 + (k-1)p} \right],$$

and this is seen to diverge to ∞ by elementary infinite product analysis. This completes the proof.

Define the auxiliary generating functions $H_n(x, t)$ by

$$H_n(x, t) = \sum_{m=1}^{\infty} \frac{C_{mn}^{(p)}}{C_{1n}^{(p)}} \left[\frac{C_{1n}^{(p)} xt}{C_{2n}^{(p)}(1 - \alpha t)} \right]^{m-1} = \sum_{m=1}^{\infty} S_{mn}^{(p)} \left[\frac{(xt)}{(1 - \alpha t)} \right]^{m-1}. \tag{2.9}$$

Taking note of (2.6), it is clear that

$$H_n(x, t) = \frac{(1 - \alpha t)^2}{C_{1n}^{(p)}} G_n\left(\frac{C_{1n}^{(p)} x}{C_{2n}^{(p)}}, t \right)$$

$$\text{for } |t| < \frac{1}{|\alpha| + \dfrac{C_{1n}^{(p)}}{C_{2n}^{(p)}}|x|}, \text{ or for } |x| < \frac{C_{2n}^{(p)}}{C_{1n}^{(p)}}\left(\frac{1}{|t|} - |\alpha| \right). \tag{2.10}$$

Then $H_n(x, t)$ is analytic in each variable separately in the regions indicated by (2.10).

Recalling that the sequence $S_{mn}^{(p)}$ decreases monotonically with n to some non-negative limit $S_m^{(p)}$, $0 < S_m^{(p)} < S_{mn}^{(p)} < 1$, let us define

$$H(x, t) = \sum_{m=1}^{\infty} S_m^{(p)} \left[\frac{(xt)}{(1 - \alpha t)} \right]^{m-1}. \tag{2.11}$$

Because of its coefficients, (2.11) converges absolutely whenever (2.9) does. For each n, (2.9) converges when the variables satisfy (2.10). Since $H(x, t)$ does not depend on n and $(C_{1n}^{(p)}/C_{2n}^{(p)}) \to 0$, $n \to \infty$, it follows that (2.11) converges for arbitrary x when t is fixed and $|t| < |\alpha|^{-1}$, and for $|t| < |\alpha|^{-1}$ when x is any fixed number. Therefore, $H(x, t)$ is entire in x and analytic in t for $|t| < |\alpha|^{-1}$. Recall that $|\alpha| < 1$. Note that we have

$$H(x, t) = \lim_{n \to \infty} H_n(x, t), \tag{2.12}$$

where the convergence is uniform on compact subsets of the admissible regions. Since $S_{1n}^{(p)} = S_{2n}^{(p)} = 1$, $n > 1$, then

$$H(x, t) = 1 + \frac{xt}{1 - \alpha t} + \dots,$$

so, in particular, $H(x, t) \not\equiv 0$.

Let f, R and r be as in Theorem 2.1. Define

$$I_n(x) = \frac{1}{2\pi i} \int_{|t| = r} \frac{f\left(-\frac{1}{t} \right) H_n(x, t)}{(1 - \alpha t)^2} \, dt \tag{2.13}$$

where x is such that (2.10) holds with $|t| = r$. That is, (2.13) is defined, and $I_n(x)$ is analytic for

$$|x| < \frac{C_{2n}^{(p)}}{C_{1n}^{(p)}} \left(\frac{1}{r} - |\alpha| \right). \tag{2.14}$$

Similarly, let

$$I(x) = \frac{1}{2\pi i} \int_{|t|=r} \frac{f\left(-\frac{1}{t}\right) H(x, t)}{(1 - \alpha t)^2} \, dt.$$

Then $I(x)$ is entire and, by (2.12), $I_n(x) \to I(x)$ uniformly on bounded sets in the plane. Since $H_n(0, t) = H(0, t) = 1$, we compute that

$$I_n(0) = I(0) = \frac{1}{2\pi i} \int_{|t|=r} \frac{f\left(-\frac{1}{t}\right)}{(1 - \alpha t)^2} \, dt = -f'(-\alpha).$$

More generally, the derivatives of $I(x)$ at $x = 0$ are given by

$$\begin{aligned} I^{(k)}(0) &= \frac{S^{(p)}_{k+1}\, k!}{2\pi i} \int_{|t|=r} \frac{f\left(-\frac{1}{t}\right)}{(1 - \alpha t)^2} \left(\frac{t}{1 - \alpha t}\right)^k dt \\ &= -S^{(p)}_{k+1} f^{(k+1)}(-\alpha), \qquad k = 0, 1, 2, \ldots . \end{aligned}$$

A similar result holds for $I_n^{(k)}(0)$, with $S^{(p)}_{k+1}$ replaced by $S^{(p)}_{k+1,n}$. Therefore, neither $I_n(x)$ nor $I(x)$ vanishes identically unless f is constant. For nonconstant f, we can find an integer $u = u(f)$ such that

$$I_n(x) = x^u J_n(x), \qquad I(x) = x^u J(x), \tag{2.15}$$

where $J_n(0) \neq 0$ and $J(0) \neq 0$. Also, there will exist a constant $\gamma_f > 0$ such that $J(x) \neq 0$ for $|x| < \gamma_f$.

THEOREM 2.2. *Let f, R and r satisfy the hypothesis of Theorem 2.1, with f nonconstant, and let $0 < \gamma < \gamma_f$. Then for all n sufficiently large $(\theta^n f)(z)$ has no zero in the disc $|z + \alpha| \leq \gamma C^{(p)}_{1n} / C^{(p)}_{2n}$.*

PROOF. On the contrary, suppose we could find a subsequence z_{n_k} such that $(\theta^{n_k} f)(z_{n_k}) = 0$ and $z_{n_k} + \alpha = -\zeta_{n_k} = -C^{(p)}_{1n_k} x_{n_k} / C^{(p)}_{2n_k}$, where $|x_{n_k}| \leq \gamma$, and where n_k is large enough that (2.14) holds for all k. Combining (2.8), (2.10), (2.13) and (2.15), there follows

$$0 = (\theta^{n_k} f)(z_{n_k}) = -C^{(p)}_{1n_k} (z_{n_k} + \alpha)^{1 + n_k p} x^u_{n_k} J_{n_k}(x_{n_k}), \tag{2.16}$$

and so $J_{n_k}(x_{n_k}) = 0$ for all k. Since $|x_{n_k}| \leq \gamma$, yet another subsequence of $\{x_{n_k}\}$ converges to a point x_0 such that $|x_0| \leq \gamma < \gamma_f$ and $J(x_0) = 0$. This contradiction proves the theorem.

REMARK. It may be that $J(x)$ has no zeros, in which case $\gamma_f = \infty$. In this situation the discs $0 < |z + \alpha| < \gamma C^{(p)}_{1n} / C^{(p)}_{2n}$, for every $\gamma > 0$, are free of zeros of $(\theta^n f)(z)$ for all n sufficiently large, depending on γ. Alternatively, $J(x)$ has zeros. If $J(x_0) = 0$, we determine by Hurwitz's Theorem a sequence of points $x_n \to x_0$ such that $J_n(x_n) = 0$. If $z_n = -\alpha - x_n(C^{(p)}_{1n} / C^{(p)}_{2n})$, then $(\theta^n f)(z_n) = 0$ by (2.16), and we also have the asymptotic relation

$$\frac{C^{(p)}_{2n}}{C^{(p)}_{1n}} (z_n + \alpha) \sim |x_0| e^{i(\pi + \arg(x_0))}, \qquad n \to \infty. \tag{2.17}$$

This is analogous to Theorem 35 of [7, pp. 172–173].

With regard to the asymptotic form of

$$C_{2n}^{(p)}/C_{1n}^{(p)} = \prod_{k=1}^{n}\left[1 + \frac{1}{1 + (k-1)p}\right],$$

a straightforward analysis shows that

$$e(1 + np)^{1/p} > C_{2n}^{(p)}/C_{1n}^{(p)} > \left(\frac{2+p}{1+p}\right)^{\log(1+np)^{1/p}}, \qquad p > 0.$$

As regards neighborhoods of points $z \neq \alpha$, we cannot say as much. Let β satisfy $|\beta| < R$, $\beta \neq -\alpha$, and let $w = T(z) = -[p\gamma(z + \alpha)^p]^{-1}$, where the branch is chosen so as to be analytic at β. The map is locally invertible, so there exists a function $F(w)$ analytic at $T(\beta)$ such that $f(z) = F(T(z)) = F(w)$. By definition of $T(z)$,

$$(\theta^n f)(z) = (D^n F)(w), \qquad n = 0, 1, 2, \ldots,$$

where D stands for ordinary differentiation. Apply the Whittaker-Gončarov theory to $F(w)$ and translate the information over to the iterates $(\theta^n f)(z)$. Unless f is a polynomial in $(z + \alpha)^{-p}$, there exists a sequence of discs D_n, shrinking with increasing n to $z = \beta$, and a subsequence $\{n_k\}$ such that $(\theta^{n_k} f)(z)$ has no zero in punctured discs D_{n_k}, $k = 1, 2, 3, \ldots$.

3. Applications. We consider zeros of successive derivatives of two classes of analytic functions, which correspond to taking $p = 1$ and $p = 0$ in Theorem 2.2.

Case I: $p = 1$. Let $F(w)$ be a function of the type considered by Boas [1] and Widder [7], that is,

$$F(w) = b_1 w^{-1} + b_2 w^{-2} + \ldots, \qquad \text{(nonconstant)}$$

analytic for $|w| > R^{-1}$, $R > 1$. Let $f(z) = F(-1/z)$, and $(\theta f)(z) = z^2 f'(z)$, so that $(\theta^n f)(z) = (D^n F)(w)$, $n = 0, 1, 2, \ldots$. By Theorem 2.2, the regions $0 < |z| < \gamma(n+1)^{-1}$ are eventually zero-free for $\gamma < \gamma_f$. Thus for $\gamma < \gamma_f$ and all n large, $F^{(n)}(w)$ has no zero which satisfies $\infty > |w| > \gamma^{-1}(n + 1)$, and this is the conclusion of Theorem A. Note that $S_m^{(1)} = 1/(m-1)!$, and so $H(x, t) = \exp(xt)$ and

$$I(x) = \frac{1}{2\pi i}\int_{|t|=r} F(t) e^{xt}\, dt.$$

That is, $I(x)$ is the inverse Laplace transform of F. Interpreting (2.17), any zero $x_0 \neq 0$ of $I(x)$ gives rise to a sequence $\{w_n\}$ of zeros of $F^{(n)}(w)$ which asymptotically approach rays (see [7, Theorem 35])

$$w_n \sim \frac{(n+1)e^{-i\arg(x_0)}}{|x_0|}, \qquad n \to \infty.$$

Case II: $p = 0$. Let $F(w)$ be a function of the form $F(w) = f(e^w)$, where $f(z)$ is analytic in $|z| < R$, $R > 1$. Then $f(z) = F(\ln z)$, and $F(w)$ is analytic in the half-plane $\mathrm{Re}(w) < \ln R$, periodic in the imaginary direction, and tends uniformly to a limit as $\mathrm{Re}(w) \to -\infty$. Define θ by $\theta = z\, d/dz$. Then with $f(z) = F(\ln z)$, we have $(\theta^n f)(z) = (D^n f)(w)$. Theorem 2.2 asserts that constants $\gamma > 0$ exist for which the discs $0 < |z| < \gamma 2^{-n}$ contain no zeros of $(\theta^n f)(z)$ for all n sufficiently large.

Equivalently, the region $\mathrm{Re}(w) < \ln \gamma - n \ln 2$ is free of zeros of $F^{(n)}(w)$. To the zeros of $J(x)$ correspond horizontal lines, instead of rays from the origin. If $J(x_0) = 0$, then there exists a sequence $\{w_n\}$ such that $F^{(n)}(w_n) = 0$ and

$$w_n \sim \ln|x_0| - n \ln 2 + i(\pi + \arg(x_0)), \qquad n \to \infty.$$

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