SEPARATING TWO COMPACT SETS BY A PARALLELOTOPE

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ABSTRACT. Let P and Q be compact subsets of Euclidean n -space. Paul Kirchberger has established that there exists a closed half-space M such that $P \subset M$ and $Q \cap M = \emptyset$ iff for each set S consisting of $n + 2$ or fewer points of $P \cup Q$, there exists a closed half-space M_S such that $(P \cap S) \subset M_S$ and $(Q \cap S) \cap M_S = \emptyset$. In this paper the problem of replacing the half-spaces by parallelotopes is considered, and the critical number of points in $P \cup Q$ is shown to be $n + 1$. Applications of this are drawn to systems of linear inequalities and to Carathéodory's theorem.

The question of separating two compact sets by various geometric figures has been of interest since the turn of the century. The first combinatorial result in this direction was established by Paul Kirchberger [2] in 1903 when he proved (in a slightly different form): "Suppose P and Q are nonempty compact subsets of Euclidean space E^n . Then there exists a closed half-space M such that $P \subset M$ and $Q \cap M = \emptyset$ iff for each set S consisting of $n + 2$ or fewer points of $P \cup Q$, there exists a closed half-space M_S such that $(P \cap S) \subset M_S$ and $(Q \cap S) \cap M_S = \emptyset$."

The problem of replacing the half-spaces by other geometric figures was posed by F. A. Valentine [5] in 1964. For results pertaining to spheres and k -cylinders, see Lay ([3], [4]). (A k -cylinder is the algebraic sum of a compact convex set and a k -dimensional subspace.) In this paper we shall investigate the problem of replacing the half-spaces by parallelotopes. This type of separation is particularly useful because the separating set can be described by a system of simultaneous linear inequalities. It also enables us to derive a strengthened form of the classical Carathéodory Theorem.

1. Definitions and notation. Given a basis $\beta \equiv \{b_1, b_2, \dots, b_n\}$ for E^n , let H_i ($i = 1, \dots, n$) be the $(n - 1)$ -dimensional subspace spanned by $\beta \sim \{b_i\}$. A β -box is a (possibly degenerate) compact solid parallelotope in which each side is parallel to one of the H_i ($i = 1, \dots, n$).

Algebraically, for any point $x \equiv \alpha_1 b_1 + \dots + \alpha_n b_n$ we define the i th coordinate function π_i by $\pi_i(x) = \alpha_i$. Then the null space of π_i is H_i and a β -box is the set of all points x that satisfy a system of linear inequalities of the form

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$$\begin{aligned}
m_1 &< \pi_1(x) < M_1 \\
m_2 &< \pi_2(x) < M_2 \quad \text{for real numbers } m_i < M_i \ (i = 1, \dots, n). \\
&\vdots \\
m_n &< \pi_n(x) < M_n
\end{aligned}$$

Given any nonempty compact subset P of E^n there is a unique minimal β -box containing P . Specifically, take

$$m_i = \inf_{x \in P} \pi_i(x) \quad \text{and} \quad M_i = \sup_{x \in P} \pi_i(x) \quad (i = 1, \dots, n).$$

The β -box so determined clearly contains P and is contained in any other β -box containing P .

2. THEOREM. *Let P and Q be nonempty compact subsets of E^n ($n \geq 2$). Then for any basis β of E^n the following three conditions are equivalent:*

- (1) *There exists a β -box B such that $P \subset B$ and $Q \cap B = \emptyset$.*
- (2) *For each subset S of $n + 1$ or fewer points of $P \cup Q$, there exists a β -box B_S such that $(P \cap S) \subset B_S$ and $(Q \cap S) \cap B_S = \emptyset$.*
- (3) *For each subset T of n or fewer points of P , the minimal β -box containing T is disjoint from Q .*

PROOF. The implication (1) $\Rightarrow$ (2) is obvious. To see that (2) implies (3), suppose that (3) does not hold. Then there exists a subset T of n or fewer points of P such that Q intersects the minimal β -box B_T containing T . Let $q \in Q \cap B_T$. Then $S \equiv \{q\} \cup T$ is a subset of $n + 1$ or fewer points of $P \cup Q$ such that there is no β -box containing $P \cap S = T$ and disjoint from $Q \cap S = \{q\}$. Thus condition (2) does not hold and we have proved the contrapositive of (2) $\Rightarrow$ (3).

The proof that (3) implies (1) is by an induction on the dimension n of the space. Suppose that $n > 2$ and that the theorem holds for spaces of dimension m , where $2 \leq m < n$. Suppose further that condition (1) does not hold. Then the minimal β -box B containing P is not disjoint from Q . That is, there exists a point q in $Q \cap B$. We must show that condition (3) does not hold.

Define a projection mapping f from E^n onto E^{n-1} by

$$f(\alpha_1 b_1 + \alpha_2 b_2 + \dots + \alpha_n b_n) = \alpha_2 b_2 + \dots + \alpha_n b_n.$$

Then $\beta' \equiv \{b_2, \dots, b_n\}$ is a basis for E^{n-1} and $f(B)$ is the minimal β' -box containing $f(P)$. Now $f(q) \in f(B)$, so the theorem applied to the compact sets $f(P)$ and $f(q)$ in E^{n-1} implies there exists a subset T' of $n - 1$ or fewer points of $f(P)$ such that $f(q)$ is in the minimal β' -box $B_{T'}$ containing T' . Let T be a subset of $n - 1$ or fewer points of P such that $f(T) = T'$. Since $f(q) \in B_{T'}$ and $\pi_i \circ f = \pi_i$ for $i = 2, \dots, n$, it follows that

$$\inf_{x \in T} \pi_i(x) < \pi_i(q) < \sup_{x \in T} \pi_i(x)$$

for $i = 2, \dots, n$. If these inequalities also hold for $i = 1$, then q is in the minimal β -box B_T containing T , and we are done. If they do not hold for $i = 1$, then either

$$\pi_1(q) < \inf_{x \in T} \pi_1(x) \quad \text{or} \quad \pi_1(q) > \sup_{x \in T} \pi_1(x).$$

If $\pi_1(q) < \inf_{x \in T} \pi_1(x)$, then since P is compact we may choose a point p in P such that $\pi_1(p) = \inf_{x \in P} \pi_1(x)$. Since $q \in B$ we have $\pi_1(q) \geq \inf_{x \in P} \pi_1(x)$ and it follows that

$$\inf_{x \in T \cup \{p\}} \pi_1(x) \leq \pi_1(q) \leq \sup_{x \in T \cup \{p\}} \pi_1(x).$$

Similarly, if $\pi_1(q) > \sup_{x \in T} \pi_1(x)$ we choose a point p in P such that $\pi_1(p) = \sup_{x \in P} \pi_1(x)$ and obtain the same result. Thus in both cases we have found a subset of n or fewer points of P whose minimal β -box contains the point q . Therefore condition (3) does not hold and we have established the induction part of the proof.

FIGURE 1

Finally we must show that (3) implies (1) when $n = 2$. Assume that condition (3) holds and let B be the minimal β -box containing P . Since B is minimal we may select a point of P on each of the four sides of the parallelogram B . Call these points p_1, p_2, p_3 , and p_4 . (They are not necessarily unique or distinct.) Suppose that there exists a point q in $Q \cap B$. We must find two points of P such that q lies in the minimal β -box containing those two points. Now for each pair of points p_i and p_j ($i, j = 1, 2, 3, 4$), let B_{ij} be the minimal β -box containing $\{p_i, p_j\}$. Then it is easy to see that B is the union of all the B_{ij} 's. In fact, B can be expressed as the union of five or fewer of the B_{ij} 's. For example, see Figure 2 below. Thus $q \in B_{ij}$ for some particular i and j . This contradicts our assumption that condition (3) holds and so Q and B must be disjoint.

$$B = B_{12} \cup B_{14} \cup B_{23} \cup B_{24} \cup B_{34}$$

FIGURE 2

3. Applications. If we rephrase the theorem in terms of systems of inequalities we obtain the following corollary.

COROLLARY 1. *Let P be a nonempty compact subset of E^n ($n \geq 2$) and suppose that a point q in E^n satisfies the following system of inequalities:*

$$\inf_{x \in P} \pi_i(x) \leq \pi_i(q) \leq \sup_{x \in P} \pi_i(x) \quad \text{for } i = 1, \dots, n.$$

Then there exists a subset T of n or fewer points of P such that

$$\inf_{x \in T} \pi_i(x) \leq \pi_i(q) \leq \sup_{x \in T} \pi_i(x) \quad \text{for } i = 1, \dots, n.$$

It is not surprising in Corollary 1 that there exists a finite subset T with the desired property. Indeed, we can choose a maximizing point and a minimizing point in P for each coordinate and readily obtain a subset T of $2n$ points in P which work. The force of the corollary is that there exist n points in P with the same property.

There is a special case included in the theorem which is of particular interest because of its similarity to the classical theorem of Carathéodory [1]:

“Let P be a compact subset of E^n and let q be a point in E^n . Suppose that given any subset T of $n + 1$ or fewer points of P there exists a hyperplane which strictly separates T and q . Then q is not in the convex hull of P .”

In the statement of this theorem we understand that a hyperplane H is said to strictly separate T and q if T lies in one of the open half-spaces determined by H and q lies in the other. The following corollary shows that if the orientation of the separating hyperplanes is properly restricted, then it suffices to consider subsets of n or fewer points of P .

COROLLARY 2. *Let P be a nonempty compact subset of E^n ($n \geq 2$) and let q be a point in E^n . Given a basis $\beta \equiv \{b_1, \dots, b_n\}$ of E^n , let H_i ($i = 1, \dots, n$) be the $(n - 1)$ -dimensional subspace spanned by $\beta \sim \{b_i\}$. Suppose that given any subset T of n or fewer points of P there exists a hyperplane which is parallel to one of the H_i ($i = 1, \dots, n$) and which strictly separates T and q . Then q is not in the convex hull of P .*

PROOF. The hypotheses imply that for each subset T of n or fewer points of P there exists a β -box containing T and disjoint from $\{q\}$. Applying the theorem we conclude that there exists a β -box containing P and disjoint from $\{q\}$, and so q is not in the convex hull of P .

4. Remarks. In the proof of the main theorem, it is crucial that all of the parallelotopes are similarly oriented. What happens if this restriction is dropped? It turns out that under these conditions there is no Kirchberger-type number. Specifically, given any positive integer k , there exist nonempty compact subsets P and Q of E^2 with the following property:

Given any subset T of k or fewer points of P , there exists a parallelogram containing T which is disjoint from Q . But there is no parallelogram containing all of P which is disjoint from Q .

We may construct P and Q in the following way.

Let m be an even number $\geq k$ and let P be the vertices of a regular $(m+1)$ -gon centered at the origin. (See Figure 3 for $m = 6$.) Suppose one point, say x , of P is omitted. Let R_1 be a circumscribing rectangle around the remaining points such that one side of R_1 contains the edge of the original polygon which is opposite to x . Let r_1 be the distance from the origin to the farthest corner (point b) of R_1 .

Now let R_2 be a rectangle circumscribing all of P which is oriented similarly to R_1 . Let r_2 be the distance from the origin to the farthest corner (point c) of R_2 . Then it is clear that $r_2 > r_1$. (The two triangles $\overline{oxc}$ and $\overline{oab}$ have equal bases $\overline{ab}$ and $\overline{xc}$. Since the altitude of the former triangle is longer than the latter's, so is its hypotenuse.) Furthermore, from symmetry considerations it follows that any parallelogram circumscribing P will have a corner at least as far as r_2 from the origin. If r is a positive number satisfying

$$r_1 < r < r_2,$$

then the circle Q of radius r centered at the origin will then have the required properties.

FIGURE 3

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