VANISHING OF THE ROCHLIN INVARIANTS OF SOME Z_2 -HOMOLOGY 3-SPHERES

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ABSTRACT. A Z_2 -homology 3-sphere has the Rochlin invariant (= μ -invariant) zero if it admits an orientation-reversing, piecewise-linear autohomeomorphism of finite order.

Throughout this paper, spaces and maps will be considered in the piecewise-linear category unless otherwise stated. The purpose of this paper is to prove the following theorem:

THEOREM. *Suppose a Z_2 -homology 3-sphere S admits an orientation-reversing autohomeomorphism h of finite order (i.e., $h^n = \text{identity}$ for some n). Then the Rochlin invariant (= μ -invariant) $\mu(S)$ of S is zero.*

This was conjectured by Joan S. Birman in her private communication of September 30, 1978, to the author, at least for Z -homology 3-spheres. This vanishing theorem is motivated by a simplicial triangulation problem [that is, if a Z -homology 3-sphere S with $\mu(S) \neq 0$ admits an orientation-reversing autohomeomorphism, then all topological manifolds (without boundary) of dimension > 5 would be simplicial complexes]. (Cf. T. Matumoto [14], D. Galewski-R. Stern [13].) In case h is an involution, our theorem is known by J. S. Birman [1], P. Pao-W. C. Hsiang [10], D. Galewski-R. Stern [4] and the author [6]. After having submitted the first draft of this paper, the author has received a paper [12] from L. Contreras-Caballero (at Cambridge) where the same result as above is obtained in the case of Z -homology 3-spheres. An improvement of her proof to the case of Z_2 -homology 3-spheres is possible [in fact, the author has done it²], but it does not seem to be simple. Recently, L. C. Siebenmann [15] uses the above vanishing theorem to prove that $\mu(S) = 0$, if S is a prime and sufficiently large or Seifert or hyperbolic Z_2 -homology sphere or a connected sum of such manifolds, admitting an orientation-reversing diffeomorphism. Thus, the above vanishing theorem has a basic importance. The proof of our theorem depends heavily on the following proposition (using results of L. H. Kauffman [5] on a branched cyclic cover of a knot) and arguments of [6] (containing a concept of the Arf invariant of a Z_2 -homology handle).

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Let k be a knot in $S^3 = \partial D^4$ with a connected Seifert surface F . (D^4 is a 4-disk.) Let $S(k)_n$ be the n -fold branched cyclic cover of S^3 along k . Let $D(F)_n$ be the n -fold branched cyclic cover of D^4 along a proper surface obtained from F by pushing $\text{Int } F$ into $\text{Int } D^4$.

PROPOSITION. $D(F)_n$ is a compact, simply connected, spin 4-manifold with boundary $S(k)_n$. If the knot k is algebraically slice and $H_1(S(k)_n; \mathbb{Q}) = 0$, then the signature $\sigma(D(F)_n) = 0$.

REMARK. In case n is a power of a prime p , then $S(k)_n$ is a $\mathbb{Z}_p$ -homology 3-sphere and hence $\mathbb{Q}$ -homology 3-sphere. [PROOF. Note that $H_1((S^3 - k)_n; \mathbb{Z}) = \mathbb{Z} \oplus H_1(S(k)_n; \mathbb{Z})$ where $(S^3 - k)_n$ is the n -fold cyclic connected cover of $S^3 - k$. If n is a power of p , then by the Wang exact sequence with $\mathbb{Z}_p$ coefficients we have that $H_1((S^3 - k)_n; \mathbb{Z}_p) = \mathbb{Z}_p$ (cf. [6, Lemma 4.2]).]

A closed (possibly nonorientable) 3-manifold M is a $\mathbb{Z}_2$ -homology handle if $H_*(M; \mathbb{Z}_2) = H_*(S^1 \times S^2; \mathbb{Z}_2)$ and $H_1(M; \mathbb{Z})$ is infinite. In [6, §4], we have defined an invariant of the homeomorphism type of a $\mathbb{Z}_2$ -homology handle M , taking an integer mod 2 and denoted by $\epsilon(M)$, as follows: Let F be a closed connected orientable surface in M transversal to a circle representing a generator of $H_1(M; \mathbb{Z})/\text{odd torsion} \approx \mathbb{Z}$. We thicken F to an imbedding $F \times [0, 1] \subset M$. Given $x \in H_1(F; \mathbb{Z}_2)$, then set $q(x) = \text{mod } 2$ linking number of $x \times 0$ and $x \times 1$ in M . The map $q: H_1(F; \mathbb{Z}_2) \rightarrow \mathbb{Z}_2$ is a quadratic form mod 2, i.e., $q(x + y) = q(x) + q(y) + x \cdot y$. We define $\epsilon(M)$ to be the Arf invariant of q . We say that $\epsilon(M)$ is the Arf invariant of the $\mathbb{Z}_2$ -homology handle M .

The Rochlin invariant (= μ -invariant) of a $\mathbb{Z}_2$ -homology 3-sphere S , denoted by $\mu(S)$, is defined to be $\mu(S) = \sigma(W)/16 \in \mathbb{Q}/\mathbb{Z}$ for any spin (i.e., $w_1 = w_2 = 0$) 4-manifold W with $\partial W = S$. As is easily seen, $\mu(S)$ is independent of a particular choice of W by Rochlin's theorem. (See V. A. Rochlin [11], M. Freedman-R. Kirby [3] for a geometric proof and Y. Matsumoto [9] for an elementary proof.)

1. Proof of proposition. Clearly $D(F)_n$ is a compact, connected, orientable 4-manifold with boundary $S(k)_n$. By [5, Lemma 5.3], $D(F)_n$ is simply connected. By [5, Corollary 5.7], the intersection pairing on $H_2(D(F)_n; \mathbb{Z})$ over $\mathbb{Z}$ is represented by a matrix $A_n + A'_n$ with respect to a suitable basis of $H_2(D(F)_n; \mathbb{Z})$ (which is free), where A'_n is the transpose of A_n . From this we see that the intersection number $x \cdot x = 0 \pmod{2}$ for $x \in H_2(D(F)_n; \mathbb{Z})$. By the Wu formula this implies that $w_2(D(F)_n) = 0$, i.e., $D(F)_n$ is spin. To prove the latter half of the proposition, we need further information on the matrix A_n . We thicken the Seifert surface F of k to an imbedding $F \times [0, 1] \subset S^3$. The Seifert form $\theta: H_1(F; \mathbb{Z}) \times H_1(F; \mathbb{Z}) \rightarrow \mathbb{Z}$ is defined by $\theta(x, y) = \text{linking number of } x \times 0 \text{ and } y \times 1 \text{ in } S^3$ for $x, y \in H_1(F; \mathbb{Z})$. The knot k is algebraically slice if $\theta(x_i, x_j) = 0$, all i, j , for a half basis $\{x_i\}$ of $H_1(F; \mathbb{Z})$. Note that this vanishing property is independent of a particular choice of Seifert surfaces of k . (Cf. J. Levine [7, Lemmas 1 and 2].)

LEMMA. If k is algebraically slice, then there is a symplectic basis $x_1, \dots, x_m, y_1, \dots, y_m$ for $H_1(F; \mathbb{Z})$ such that $\theta(x_i, x_j) = x_i \cdot x_j = y_i \cdot y_j = 0$, all i, j , and $x_i \cdot y_j = \delta_{ij}$.

Since there is a canonical homomorphism from the mapping class group of F (of genus m) onto the group $S_p(2m, Z)$ of all integral symplectic matrices (see, for example, [8, Theorem N13, p. 178]), the basis $x_1, \dots, x_m, y_1, \dots, y_m$ of $H_1(F; Z)$ in the lemma is represented by a system of simple closed curves $x_1^0, \dots, x_m^0, y_1^0, \dots, y_m^0$ such that $x_i^0 \cap x_j^0 = y_i^0 \cap y_j^0 = \emptyset$, all i, j , and $x_i^0 \cap y_j^0 =$ one point (if $i = j$) or $\emptyset$ (if $i \neq j$). Then L. H. Kauffman [5, Corollary 5.7] showed that there exists a basis of $H_2(D(F)_n; Z)$ (associated with this basis $x_1, \dots, x_m, y_1, \dots, y_m$) so that the above matrix A_n is the tensor product $A \otimes L_n$ where A is the matrix representing θ with respect to the basis $x_1, \dots, x_m, y_1, \dots, y_m$ of $H_1(F; Z)$ and

$$L_n = \begin{bmatrix} 1 & -1 & & & 0 \\ & 1 & -1 & & \\ & & \ddots & \ddots & \\ & 0 & & \ddots & -1 \\ & & & & 1 \end{bmatrix}, \quad ((n-1) \times (n-1)).$$

Since $\theta(x_i, x_j) = 0$, all i, j , we can see that the intersection pairing on $H_2(D(F)_n; Z)$ over Z vanishes for a half basis of $H_2(D(F)_n; Z)$. (Cf. [5, Proposition 5.6].) Thus, if k is algebraically slice and $H_1(S(F)_n; Q) = 0$, then the intersection pairing on $H_2(D(F)_n; Q)$ over Q vanishes for a half basis of $H_2(D(F)_n; Q)$ and is nonsingular, so that the signature $\sigma(D(F)_n) = 0$. This completes the proof of the proposition except for the proof of the lemma.

PROOF OF LEMMA. Let $x_1, \dots, x_m, y'_1, \dots, y'_m$ be a basis for $H_1(F; Z)$ such that $\theta(x_i, x_j) = 0$, all i, j . Note that $x_i \cdot x_j = 0$, all i, j , for $a \cdot b = \theta(a, b) - \theta(b, a)$. Let x_1^* be an element of $H_1(F; Z)$ with $x_i \cdot x_1^* = \delta_{i1}$, $y'_j \cdot x_1^* = 0$. [Use the nonsingularity of the intersection pairing on $H_1(F; Z)$.] $H_1(F; Z)$ has an orthogonal splitting $\{x_1, x_1^*\} \oplus V$ with respect to the intersection pairing on $H_1(F; Z)$, where $\{a_1, \dots, a_c\}$ denotes the subgroup generated by $a_1, \dots, a_c$ and $V = \{x_1, x_1^*\}^\perp$. Note that $x_2, \dots, x_m$ are contained in V . Since $H_1(F; Z)/\{x_1, \dots, x_m\} = \{x_1, x_1^*\}/\{x_1\} \oplus V/\{x_2, \dots, x_m\}$ is free of rank m , $V/\{x_2, \dots, x_m\}$ is free of rank $m-1$. By induction, V admits an orthogonal splitting $\{x_2, x_2^*\} \oplus \dots \oplus \{x_m, x_m^*\}$ with respect to the restriction of the intersection pairing on $H_1(F; Z)$ to V . We take $x_1^*, \dots, x_m^*$ as $y_1, \dots, y_m$, proving the lemma.

2. Proof of theorem. First we note the following:

2.1 (BIRMAN). *If S admits an orientation-reversing autohomeomorphism h of finite order, then S admits an orientation-reversing autohomeomorphism of order 2^s for some $s > 1$.*

To see this, note that the order n of h is even, since h is orientation-reversing. Let $n = 2^s n'$ where $s > 1$ and n' is odd. $h^{n'}$ is still orientation-reversing and has order 2^s , proving 2.1.

By 2.1, we assume the order of h is 2^s , $s > 1$. If $s = 1$, we showed in [6, Theorem IV] that $\mu(S) = 0$. So, assume also that $s > 2$. Since h is orientation-reversing and $H_1(S; Q) = 0$, we see that $\text{Fix}(h, S) \neq \emptyset$ (e.g., by the Lefschetz fixed-point formula), so $\text{Fix}(h^{2^{s-1}}, S) \neq \emptyset$. Then by Smith theory (cf. [2, Chapter IV, §4], and

[6, Proposition 6.1]) $\text{Fix}(h^{2^{i-1}}, S)$ is a 1-sphere, say k , since $h^{2^{i-1}}$ is an orientation-preserving involution on S . Note that $h(k) = k$ and $h^{2^i}|_k, i > 1$, is orientation-preserving. It follows that $\text{Fix}(h^{2^{i-1}}, S) = \text{Fix}(h^{2^{i-2}}, S) = \cdots = \text{Fix}(h^2, S) = k$. Further, using $h^2 = \text{identity}$, we see that $\text{Fix}(h^{2n'}, S) = \text{Fix}(h^2, S)$ for any odd n' , so that $\text{Fix}(h^{n'}, S) = \text{Fix}(h, S) \subset \text{Fix}(h^{n''}, S) = k$ for any odd n' and even n'' with $n'' \not\equiv 0 \pmod{2^s}$. Now we let $S^* = S/h^2$ and $\rho: S \rightarrow S^*$ be the canonical projection and $\rho(k) = k^*$. Note that $\rho|_k: k \rightarrow k^*$ is an injection and S^* is a closed connected 3-manifold. We can see that S^* is a Z_2 -homology 3-sphere. [PROOF. Note that $\rho|_{S-k}: S-k \rightarrow S^*-k^*$ is a covering, so that $\rho_\#: \pi_1(S, x) \rightarrow \pi_1(S^*, \rho(x))$ is onto for $x \in k$, showing that $H_1(S^*; Z_2) = 0$, since $H_1(S; Z_2) = 0$.] S^* admits an orientation-reversing involution h^* defined by h . By Smith theory (cf. [2, Chapter IV, §4], and [6, Proposition 6.1]), $\text{Fix}(h^*, S^*) = S^0$ or S^2 , but S^2 does not occur, since $\text{Fix}(h^*, S^*) \subset k^*$. So, $\text{Fix}(h^*, S^*) = S^0 \subset k^*$. In [6, Proof of Theorem 5.1], we have constructed a 4-manifold $W_1^* = S^* \times [0, 1] \cup D^2 \times D^2$ by attaching $(\partial D^2) \times D^2$ to $S^* \times 1$ along $k^* \times 1$ so as to satisfy the following: (i) $\partial W_1^* - S^* \times 0 = M^*$ is a Z_2 -homology handle, (ii) h^* extends to an orientation-reversing involution $\bar{h}_1^*$ on W_1^* such that $\text{Fix}(\bar{h}_1^*, W_1^*)$ is a proper arc contained in the proper 2-disk $D^* = k^* \times [0, 1] \cup D^2 \times 0$, and, in particular, $h_1^* = \bar{h}_1^*|_{M^*}$ is an orientation-reversing free involution. If $\epsilon(M^*) = 0$, then by [6, Corollary 4.1], M^* bounds a spin 4-manifold W_2^* with an orientation-reversing free involution $\bar{h}_2^*$ extending h_1^* such that the canonical homomorphism $H_1(M^*; Z)/\text{odd torsion} (\approx Z) \rightarrow H_1(W_2^*; Z)/\text{odd torsion}$ is an isomorphism (cf. [6, Remark 4.2]).

2.2. If $\epsilon(M^*) = 0$, then S^* bounds a spin 4-manifold $W^* = W_1^* \cup W_2^*$ with an orientation-reversing involution $\bar{h}^*$ (defined by $\bar{h}_1^*$ and $\bar{h}_2^*$) extending h^* such that $\text{Fix}(\bar{h}^*, W^*)$ is a proper arc contained in the proper disk $D^* = k^* \times [0, 1] \cup D^2 \times 0$ and $H_1(W^*; Z_2) = 0$.

Let $W = W_1 \cup W_2$ be the 2^{s-1} -fold branched cyclic cover of W^* along D^* (i.e., associated with an epimorphism $H_1(W^* - D^*; Z)/\text{odd torsion} (\approx Z) \rightarrow Z_{2^{s-1}}$), where W_i corresponds to W_i^* , $i = 1, 2$. As an analogy of [6, proof of Corollary 4.1], we see that ∂W_2 is a Z_2 -homology handle and the canonical homomorphism $H_1(\partial W_2; Z)/\text{odd torsion} \rightarrow H_1(W_2; Z)/\text{odd torsion}$ is an isomorphism. By [6, Lemma 4.5], W_2 is spin. As is easily seen, W_1 can be written as $S \times [0, 1] \cup D^2 \times D^2$. [Note that the canonical homomorphism $H_1(S^* \times 1 - k^* \times 1; Z)/\text{odd torsion} \rightarrow H_1(M^*; Z)/\text{odd torsion} (\approx Z)$ is an isomorphism.] Since $\partial W_1 - S \times 0$ is a Z_2 -homology handle, we see that W is a spin 4-manifold. Since $\bar{h}^*$ lifts to an orientation-reversing autohomeomorphism $\bar{h}$ of order 2^s extending h , we obtain the following:

2.3. If $\epsilon(M^*) = 0$, then S bounds a spin 4-manifold W with an orientation-reversing autohomeomorphism $\bar{h}$ of order 2^s extending h and $H_1(W; Z_2) = 0$.

To consider the case $\epsilon(M^*) = 1$, we make use of some properties of the figure eight knot $k_a \subset S^3$. For this knot, there exists an orientation-reversing involution α on S^3 such that $\alpha(k_a) = k_a$ and $\text{Fix}(\alpha, S^3) = S^0 \subset k_a$. Take a knot sum $(k^* \subset S^*) \# (k_a \subset S^3) = (k_a^* \subset S_a^*)$ so as to admit an orientation-reversing involution h_a^*

defined by h^* and α with $\text{Fix}(h_a^*, S^*) = S^0 \subset k_a^*$. Let S_a be the 2^{s-1} -fold branched cyclic cover of S_a^* along k_a^* and h_a be the lift of h_a^* which is an orientation-reversing autohomeomorphism of order 2^s . Note that S_a is a Z_2 -homology 3-sphere, homeomorphic to $S \# S(k_a)_{2^{s-1}}$. As discussed in [6, proof of Theorem 5.1], the manifold M_a^* for $k_a^* \subset S^*$ in place of M^* satisfies that $\epsilon(M_a^*) = 0$ if $\epsilon(M^*) = 1$. From 2.3 we have the following:

2.4. If $\epsilon(M^*) = 1$, then $S_a = S \# S(k_a)_{2^{s-1}}$ bounds a spin 4-manifold W_a with an orientation-reversing autohomeomorphism h_a of order 2^s extending h_a and $H_1(W_a; Z_2) = 0$.

2.5. $\mu(S(k_a)_{2^i}) = 0$ for all i .

To see this, let F be a connected Seifert surface for k_a . The oriented disk sum $F \natural F$ is a connected Seifert surface for a knot sum $k_a \# k_a$ which is well known to be slice and hence algebraically slice. (Cf. J. Levine [7, Lemma 2].) Using $H_1(S(k_a \# k_a)_{2^i}; Q) = 0$, from the proposition we see that $\alpha(D(F \natural F)_{2^i}) = 0$. But, $D(F \natural F)_{2^i}$ is the oriented disk sum $D(F)_{2^i} \natural D(F)_{2^i}$. Hence $\sigma(D(F)_{2^i}) = 0$ and $\mu(S(k)_{2^i}) = \sigma(D(F)_{2^i})/16 = 0$, proving 2.5.

By 2.3 and 2.4, $\mu(S)$ is 0 or $\mu(S(k_a)_{2^{s-1}})$. By 2.5, we have always that $\mu(S) = 0$. This completes the proof of the theorem.

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