CONNECTED PROJECTIONS OF BLOCKING SETS OF $I^n \times M$

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ABSTRACT. Sufficient conditions are given for each minimal blocking set K of $I^n \times M$ to have the closure of its projection, $p(K)$, into I^n connected. The construction of some examples of almost continuous functions $f: I^n \rightarrow M$ in the literature depends on knowing each $\overline{p(K)}$ is connected or perfect.

Properties of the projection $p(K)$ into I^n of a minimal blocking set K of $I^n \times M$ have been studied in [6] and [5]. For example, in [5], Kellum proves that if M is an absolute almost continuous retract and X is a compact metric space, then $p(K)$ is connected for each minimal blocking set K of $X \times M$. In Theorems 1 and 2, we show $\overline{p(K)}$ is connected when either M is a possibly noncompact almost continuous retract of any AR like I^n or M is an arbitrary arcwise connected subset of I^n . We use this last result to obtain a corollary to a theorem of Alexander and Garrett in [1] and [2] for the case when $X = I^n$.

Throughout this paper, a function and its graph are the same. A function $f: X \rightarrow M$ is said to be *almost continuous* if in every open set of $X \times M$ containing f there is a continuous function $g: X \rightarrow M$. A subset M of X is called an *almost continuous retract* of X if there is an almost continuous function $f: X \rightarrow M$ which is the identity on M . A closed subset K of $X \times M$ is a *blocking set* if K misses some function from X into M but meets every continuous function from X into M . If $W \subset X$, we let $K|W$ denote the part of K whose X -projection lies in W .

If B is a disk in the plane R^2 and $B_1, B_2, \dots, B_n$ are disjoint disks in $\text{Int } B$, then $D = B - \bigcup \text{Int } B_i$ is called a *disk-with-holes*. Let E be another disk-with-holes such that $D \subset \text{Int } E$. E and D are said to be *concentric* if $E - \text{Int } D$ is homeomorphic to $(\text{Bd } D) \times I$.

The first theorem which follows can be used for constructing simple examples of subsets which are not almost continuous retracts of I^2 . But its converse is false because if M is a circle then it is known that M is not an almost continuous retract of I^2 [4], although according to the second theorem, $\overline{p(K)}$ is connected for each minimal blocking set K .

THEOREM 1. *If M is an almost continuous retract of I^n , then for each minimal blocking set K of $I^n \times M$, $\overline{p(K)}$ is connected.*

Received by the editors July 9, 1979 and, in revised form, September 10, 1979.

1980 *Mathematics Subject Classification.* 54C08, 54C15, 54D05; Secondary 57N05.

Key words and phrases. Blocking sets, almost continuous retracts of I^n , arcwise connected subcontinua.

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0002-9939/80/0000-0287/\$01.75

PROOF. Assume there is a minimal blocking set K of $I^n \times M$ with $\overline{p(K)} = A \cup B$ where A and B are separated. Let $r: I^n \rightarrow M$ be an almost continuous retraction. Since K is minimal, there are continuous functions $f, g: I^n \rightarrow M$ for which $p(f \cap K) \subset B$ and $p(g \cap K) \subset A$. We can extend the function $f|A \cup g|B$ to a continuous function $h: I^n \rightarrow I^n$ because I^n is an AR. Then $rh: I^n \rightarrow M$ is almost continuous [8] and therefore must meet K . But, by construction, rh misses K , which is a contradiction.

If the regular neighborhood collaring theorem [7, p. 36] is used, the following theorem (and corollary) can be proven for any n -cube I^n , $n > 2$. But for simplicity, we give the proof for $n = 2$.

THEOREM 2. *If M is an arcwise connected subset of I^2 , then for each minimal blocking set K of $I^2 \times M$, $\overline{p(K)}$ is connected.*

PROOF. Assume $\overline{p(K)} = A \cup B$ with A and B separated. First we need to construct finitely many pairwise disjoint continua $W_1, W_2, \dots, W_m$ in I^2 with $A \cup B \subset \bigcup W_i$. There are open sets $U \supset A$ and $V \supset B$ in I^2 such that $\bar{U} \cap \bar{V} = \emptyset$. The components of $U \cup V$ which meet $A \cup B$ form an open cover of $A \cup B$ that has a finite subcover $\mathcal{Q}$. Let $W_1, W_2, \dots, W_m$ denote the components of $\bigcup \mathcal{Q}$. For $i = 1, 2, \dots, m$, there are continuous functions $f_i: I^2 \rightarrow M$ with $f_i|K \cap W_i = \emptyset$. For each i , let D_i and E_i be concentric disks-with-holes with $W_i \subset \text{Int } D_i$, $D_i \subset \text{Int } E_i$, and $E_j \cap E_k = \emptyset$ whenever $j \neq k$. Since each component C_j of $\text{Bd } D_i$ bounds a disk in I^2 , each $f_i|C_j$ is inessential in M . Therefore, there is a continuous function $F: \bigcup (E_i - \text{Int } D_i) \rightarrow M$ such that $F|C_j = f_i$ and F on each component G_j of $\text{Bd } E_i$ is a constant $e_j \in M$. Without loss of generality, we may assume $e_j = e_k = e$ for $j \neq k$ because of the arcwise connectedness of M . Define $g: I^2 \rightarrow M$ by $g = \bigcup (f_i|D_i) \cup F$ on $\bigcup E_i$ and $g = e$ on $I^2 - \bigcup E_i$. Since the continuous function g misses the blocking set K , we have reached a contradiction.

Alexander and Garrett's characterization of almost continuous retracts can now be stated in the following special form.

COROLLARY. *Suppose M is an arcwise connected subcontinuum of I^2 . A necessary and sufficient condition that M be an almost continuous retract of I^2 is that each blocking set K of $I^2 \times M$, with $p(K)$ a subset of M , intersects the diagonal of $M \times M$.*

PROOF. According to [1] or [2], this is true for an arbitrary subcontinuum M of any Peano continuum X in place of I^2 if each minimal blocking set K of $X \times M$ has $p(K)$ a subset of M or $X - M$ contains an uncountable subset of $p(K)$. By Theorem 2, $(I^2 - M) \cap p(K)$ is uncountable whenever it is nonempty, and the conclusion follows.

The last result is of interest in light of Kellum's conjecture in [3] that M is an almost continuous retract of I^2 whenever M is an arcwise connected acyclic subcontinuum of I^2 .

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