PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 79, Number 3, July 1980

A NOTE ON STRONGLY E-REFLEXIVE INVERSE SEMIGROUPS

L. O'CARROLL

ABSTRACT. In contrast to the semilattice of groups case, an inverse semigroup S which is the union of strongly E-reflexive inverse subsemigroups need not be strongly E-reflexive. If, however, the union is saturated with respect to the Green's relation $\mathfrak{D}$, and in particular if the union is a disjoint one, then S is indeed strongly E-reflexive. This is established by showing that $\mathfrak{D}$-saturated inverse subsemigroups have certain pleasant properties. Finally, in contrast to the E-unitary case, it is shown that the class of strongly E-reflexive inverse semigroups is not closed under free inverse products.

The reader is referred to **[1]**, **[2]** for the basic theory of inverse semigroups, including the theory of free inverse products. Recall from **[4]**, **[5]** that an inverse semigroup S is said to be *strongly E-reflexive* whenever S is a semilattice of E-unitary inverse semigroups, or alternatively, whenever there exists a semilattice of groups congruence η on S such that only idempotents are linked to idempotents under η. In **[4]**, **[5]** we studied this class of semigroups and showed that many of the properties of semilattices of groups and of E-unitary inverse semigroups generalise to this class, albeit sometimes in a weaker form. We continue this line of investigation here.

In what is by now a classic theorem, Clifford showed that an inverse semigroup which is a union of groups is a semilattice of groups. We ask to what extent this is true for strongly E-reflexive inverse semigroups. It is already known that a semilattice of strongly E-reflexive inverse semigroups is again strongly E-reflexive **[5]**. The following simple example shows that we cannot hope for a full generalisation of Clifford's theorem.

Consider the bisimple inverse ω-semigroup $S(G, \alpha)$, where the endomorphism α of the group G is not injective. As noted in **[4**, p. 341**]**, $S(G, \alpha)$ is not strongly E-reflexive. However, using **[1**, Lemma 1.31**]**, it is easily seen that $S(G, \alpha)$ is a union of its maximal subgroups and copies of the bicyclic semigroup, and these are all E-unitary.

The restriction we require will now be given, and the example just noted would seem to indicate that it is the weakest possible.

Let S be an inverse semigroup with semilattice of idempotents E. Let U be an inverse subsemigroup of S which is $\mathfrak{D}$*-saturated* in the sense that $x \mathfrak{D} y \in U$ implies $x \in U$, where $\mathfrak{D}$ denotes the usual Green's relation on S. The maximal group homomorphic image of U is denoted by $\bar{U}$ with $\bar{u}$ denoting the image of u ($u \in U$). Let $U' = \{x \in S | x \geqslant u \text{ for some } u \in U\}$; note that U' may equal S.

Received by the editors March 23, 1979.
AMS (MOS) subject classifications (1970). Primary 20M10.

The first result shows that U' has some pleasant properties.

PROPOSITION. (i) *U' is an inverse subsemigroup of S which contains U, and $xy \in U'$ implies $x \in U'$ and $y \in U'$.*

(ii) *The rule: $x\phi = \bar{u}$ if $x \geq u \in U$ and $x\phi = 0$ otherwise, gives a well-defined homomorphism ϕ: $S \to \bar{U}^0$ such that $\phi|U$ is the canonical homomorphism onto $\bar{U}$.*

PROOF. (i) $xy \geq u \in U \Rightarrow xx^{-1} \geq xyy^{-1}x^{-1} \geq uu^{-1} \Rightarrow x \geq uu^{-1}x \ \mathcal{R}\ u \Rightarrow x \in U'$, since U is $\mathcal{D}$-saturated and $\mathcal{R} \subseteq \mathcal{D}$.

Dually, $y \in U'$. The remainder of the result is easily proven.

(ii) Suppose $x \in U'$ with $x \geq u \in U$ and $x \geq v \in U$. Then $u = ex, v = fx$ where $e = uu^{-1} \in U \cap E, f = vv^{-1} \in U \cap E$. Hence $efu = efv$, and $ef \in E \cap U$, so that $\bar{u} = \bar{v}$. It is then almost immediate that ϕ is well-defined. The rest of the result involves a little routine calculation, using (i).

REMARK. Taking S to be a semilattice with more than two elements, we see that U need not be an ideal of U' in Proposition 1.

The proposition enables us to prove our main result.

THEOREM. *Let S be a union of $\mathcal{D}$-saturated strongly E-reflexive inverse subsemigroups S_i, $i \in I$. Then S is strongly E-reflexive.*

PROOF. Each S_i is a semilattice Λ_i of E-unitary inverse semigroups T_i^λ, $\lambda \in \Lambda_i$. It is easily shown that each T_i^λ is $\mathcal{D}$-saturated in S. Hence we may suppose without loss of generality that each S_i is E-unitary. For each $i \in I$, let ϕ_i: $S \to \bar{S}_i^0$ be the homomorphism defined as in (ii) above, and let T be the direct product of the $\bar{S}_i^0$. Then the ϕ_i induce a homomorphism ϕ: $S \to T$ with $s\phi$ having ith component $s\phi_i$, $i \in I$. Now $S\phi$ is a semilattice of groups, since T is. Suppose that $x\phi = e\phi$ for some $e \in E$, where $x \in S_i$ say. Then $x\phi_i$ is the identity element of $\bar{S}_i$, and since S_i is E-unitary it follows that $x \in E$; whence the result.

COROLLARY. *Let S be a disjoint union of strongly E-reflexive inverse subsemigroups. Then S itself is strongly E-reflexive.*

PROOF. Clearly each of the inverse subsemigroups in question is $\mathcal{D}$-saturated in S.

REMARK. The elementary theory of inverse semigroups shows that an inverse semigroup S which is a union of groups is a disjoint union of its maximal subgroups H_e, $e \in E$, and that this is the $\mathcal{D}$-decomposition of S. Hence S is the union of the $\mathcal{D}$-saturated E-unitary inverse subsemigroups H_e. It is easy to show that the homomorphism ϕ in the proof of the theorem is injective in this case. Hence S is a subdirect product of the H_e with zero added possibly. From this one can deduce, again by elementary means, that S is a semilattice of groups with the multiplication defined by linking homomorphisms. Thus, modulo some elementary results, our theory restricts to Clifford's classic theorems.

Now let E be the semilattice $\{e, f, g\}$ where $e > g, f > g$, and e, f are incomparable. Let S be the semilattice of groups $G_e \cup G_f \cup G_g$ where G_e, G_g are trivial and G_f is the cyclic 2-group; let T be the semilattice of groups $H_e \cup H_f \cup H_g$

where H_g is the trivial group and H_e, H_f are copies of the cyclic 2-group (the multiplications being defined in the obvious way). Consider the word $w = eabc$ in the free inverse product P of S and T, where e is the identity element of G_e and $a[b, c]$ is the non-identity element of $H_e[G_f, H_f]$. If ψ is a semilattice of groups homomorphism on P, it is easily seen that $w\psi$ is an idempotent. On the other hand one can find a representation of S and T in $\mathcal{I}_5$, the symmetric inverse semigroup on five symbols, in which the image of w is not an idempotent. Hence w is not an idempotent, so that P is not strongly E-reflexive.

On the other hand McAlister [3] has shown that the free inverse product of two E-unitary inverse semigroups is again E-unitary.

REFERENCES

1. A. H. Clifford and G. B. Preston, *The algebraic theory of semigroups*, vols. 1, 2, Math. Surveys No. 7, Amer. Math. Soc., Providence, R. I., 1961 and 1967.

2. J. M. Howie, *An introduction to semigroup theory*, Academic Press, London, 1976.

3. D. B. McAlister, *Inverse semigroups generated by a pair of subgroups*, Proc. Roy. Soc. Edinburgh Sect. A. 77 (1977), 9–22.

4. L. O'Carroll, *Strongly E-reflexive inverse semigroups*, Proc. Edinburgh Math. Soc. **20** (1976–77), 339–354.

5. ______, *Strongly E-reflexive inverse semigroups*. II, Proc. Edinburgh Math. Soc. **21** (1978), 1–10.

DEPARTMENT OF MATHEMATICS, THE KING'S BUILDINGS, EDINBURGH EH9 3JZ, SCOTLAND