THE COMPUTATION OF $|W_*(\pi, \omega; Z)|$ FOR π AN ABELIAN 2-GROUP

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ABSTRACT. Let π be a finite abelian 2-group and $\omega: \pi \rightarrow Z_2 = \{+1, -1\}$ be a nontrivial homomorphism. Under these conditions, we compute the group $W_*(\pi, \omega; Z)$. We also show that $W_2(\pi, \omega; Z)$ is isomorphic to $W_2(\pi, \omega; Z[\frac{1}{2}])$.

In [3], we introduced the equivariant Witt theory $W_*(G, \omega; R)$, which provides invariants of equivariant bordism classes of closed G -manifolds wherein elements of G preserve or reverse orientation in a prescribed way. The advantage that this theory enjoys over its antecedents is, of course, the much greater flexibility gained in being able to consider situations where elements of G reverse orientation. It was not initially clear, however, just how difficult it might be to compute $W_*(G, \omega; Z)$ in a situation where some of the real generality of this apparatus is realized. This short note reports just such a computation for G a finite abelian 2-group, in which case we replace G by π . The brevity of this note should be read as an indicator of how superficial the potential difficulties turn out to be. Our standing assumption will be that the homomorphism $\omega: \pi \rightarrow Z_2 = \{1, -1\}$ is nontrivial, the other case having been treated in [2].

To cite our results, let π be a given finite abelian 2-group together with a homomorphism $\omega: \pi \rightarrow Z_2 = \{1, -1\}$. We can decompose π into the direct sum

$$\pi \cong Z_{2^{k_1}} \oplus \cdots \oplus Z_{2^{k_r}} \oplus Z_{2^{m_1}} \oplus \cdots \oplus Z_{2^{m_s}}, \quad (1)$$

with the property that ω is nontrivial on each of the generators of $Z_{2^{k_1}} \oplus \cdots \oplus Z_{2^{k_r}} = \pi^-$ and, furthermore, π^- is a maximal subgroup of π with respect to this property. A direct consequence of this is that ω is trivial on $Z_{2^{m_1}} \oplus \cdots \oplus Z_{2^{m_s}} = \pi^+$ and that the summands in π^- and π^+ , respectively, can be arranged so that

$$k_1 > k_2 > \cdots > k_r > m_1 > m_2 > \cdots > m_s.$$

We do not exclude the case where π^+ is the trivial group; there $s = 0$. We can now mention our results.

THEOREM 1. *Let $m_0 = k_r > 2$ and $m_{s+1} = -1$. Then the dimension of $W_0(\pi, \omega; Z)$ as a vector space over Z_2 is given by $[2^r \sum_{i=0}^s (m_i - m_{i+1}) 2^i \prod_{j=i+1}^{s+1} 2^{m_j}] - 1$. If $k_r = 1$, then $W_0(\pi, \omega; Z) = \{0\}$.*

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THEOREM 2. (a) $W_2(\pi, \omega; Z) \cong W_2(\pi, \omega; Z[\frac{1}{2}])$.

(b) If $k_1 \geq 2$, $m_0 = k_r > 1$ and $m_{s+1} = 2$, then the dimension of $W_2(\pi, \omega; Z)$ as a vector space over Z_2 is given by $2^{r-3} \sum_{i=0}^s (m_i - m_{i+1}) 2^i \prod_{j=i+1}^s 2^{m_j}$. If $k_1 < 2$ or $k_r = 1$, then $W_2(\pi, \omega; Z) = \{0\}$.

We note that (a) of Theorem 2 and Theorem 2.12 of [3] give us a complete generalization of Lemma 1.1 [1] to the theory $W_*(\pi, \omega; Z)$ for π a finite abelian p -group. Of course, when p is an odd prime, the result comes trivially as the theories $W_*(\pi, \omega; Z)$ and $W_*(\pi; Z)$ coincide.

1. Proof of Theorem 1. Let π be a finite abelian 2-group and $\omega: \pi \rightarrow Z_2 = \{1, -1\}$ be a homomorphism. Further, let π be decomposed relative to ω as in (1). We first note that if $k_r = 1$, then the proof of Lemma 2.2 [3], with the obvious modifications, applies to show that $W_*(\pi, \omega; Z[\frac{1}{2}]) = W_*(\pi, \omega; Z) = \{0\}$. Thus we may assume $k_r \geq 2$ and, letting T denote the generator of the summand $Z_{2^{k_1}}$, $\omega(T^{2^{k_1-1}}) = 1$. Consequently, $T^{2^{k_1-1}}$ acts as an isometry. Now let (V, π) represent a given class in $W_*(\pi, \omega; Z[\frac{1}{2}])$ and form the eigenspaces

$$\Sigma_{T^{2^{k_1-1}}}^+ = \{x \in V | T^{2^{k_1-1}}(x) = x\},$$

$$\Sigma_{T^{2^{k_1-1}}}^- = \{x \in V | T^{2^{k_1-1}}(x) = -x\}.$$

Using the fact that $T^{2^{k_1-1}}$ acts as an isometry, this decomposition into eigenspaces gives us the isomorphism

$$\begin{aligned} W_*(\pi, \omega; Z[\tfrac{1}{2}]) \\ \rightarrow W_*(Z_{2^{k_1-1}} \oplus Z_{2^{k_2}} \oplus \cdots \oplus Z_{2^{k_r}} \oplus Z_{2^{m_1}} \oplus \cdots \oplus Z_{2^{m_s}}, \omega; Z[\tfrac{1}{2}]) \\ \oplus W_*^-(\pi, \omega; Z[\tfrac{1}{2}]), \end{aligned} \quad (2)$$

where the latter summand consists of those inner product spaces with π -action of the form $(\Sigma_{T^{2^{k_1-1}}}^-, \pi)$. Following the pattern of [2] and [3], we now concentrate on $W_0^-(\pi, \omega; Z[\frac{1}{2}])$ in the hope of establishing an inductive procedure. We let

$$D_{k_1} = \frac{Z[\frac{1}{2}][X]}{X^{2^{k_1-1}} + 1}$$

and, as in [2] and [3], view $W_0^-(\pi, \omega; Z[\frac{1}{2}])$ as $W_0(Z_{2^{k_2}} \oplus \cdots \oplus Z_{2^{k_r}} \oplus Z_{2^{m_1}} \oplus \cdots \oplus Z_{2^{m_s}}, \omega; D_{k_1})$ where $Z_{2^{k_2}} \oplus \cdots \oplus Z_{2^{k_r}} \oplus Z_{2^{m_1}} \oplus \cdots \oplus Z_{2^{m_s}}$ acts as a group of D_{k_1} -module automorphisms, ω plays its usual role, and the inner products continue to take their values in $Z[\frac{1}{2}]$. The advantage of this construction is that we now can continue the process and break up the action of $Z_{2^{k_2}}$ on a given D_{k_1} -module into an orthogonal direct sum of D_{k_1} -modules on which $Z_{2^{k_2}}$ acts as multiplication by the various 2^{k_2} th roots of unity. This is, however, incompatible with the role of ω unless $k_2 = k_1$ and then we can only consider the two primitive 2^{k_2} th roots of unity. Neither of these restrictions carries over into our considerations of the summands $Z_{2^{m_j}}$ and indeed all of the 2^{m_j} th roots of unity may operate. From all of this we can infer, in the manner of [2] and [3], that

$$W_0^-(\pi, \omega; Z[\tfrac{1}{2}]) \cong ([H(D_{k_r})]^{2^{r-1}})^{2^{m_1+\dots+m_s}}$$

and, in turn, that $W_0(\pi, \omega; Z[\tfrac{1}{2}])$ is isomorphic to

$$\sum_{i=m_1+1}^{k_r} [H(D_i)]^{2^{m_1+\dots+m_s+r-1}} \oplus \sum_{i=m_2+1}^{m_1} [H(D_i)]^{2^{m_2+\dots+m_s+r}} \\ \oplus \dots \oplus \sum_{i=m_s+1}^{m_{s-1}} [H(D_i)]^{2^{m_s+r+s-2}} \oplus \sum_{i=2}^{m_s} [H(D_i)]^{2^{r+s-1}}.$$

Here $H(D_i)$ refers to the Witt group of D_i -valued Hermitian inner products on finitely generated projective D_i -modules (see [3, §2]). In arriving at the above computation, the reader should note that the number of summands in the maximal subgroup (on whose generators ω is nontrivial) increases as we proceed through the inductive process. To continue the computation, we recall that the groups $H(D_i)$ have been computed in [3], where it is shown that

$$H(D_i) \cong \begin{cases} Z_2 \oplus Z_2 \oplus Z_2 & \text{if } i = 2, \\ Z_2 & \text{if } i > 2. \end{cases}$$

The dimension of $W_0(\pi, \omega; Z[\tfrac{1}{2}])$ can now be recorded as

$$(k_r - m_1)2^{m_1+\dots+m_s+r-1} + (m_1 - m_2)2^{m_2+\dots+m_s+r} \\ + \dots + (m_{s-1} - m_s)2^{m_s+r+s-2} + (m_s - 2)2^{r+s-1} + 3(2^{r+s-1}). \quad (3)$$

By agreeing to let $m_0 = k_r$ and $m_{s+1} = -1$, we can reformulate this rather lengthy expression as

$$2^r \sum_{i=0}^s (m_i - m_{i+1}) 2^i \prod_{j=i+1}^{s+1} 2^{m_j}. \quad (4)$$

Now, with the application of Theorem 2.12 of [3], the proof of Theorem 1 is complete.

2. Proof of Theorem 2. We now turn to the consideration of $W_2(\pi, \omega; Z)$. Referring to the decomposition (1) of π , we know that $W_2(\pi, \omega; Z[\tfrac{1}{2}]) \neq \{0\}$ only if $k_1 > k_2 > k_r > 2$, as noted at the beginning of §1. In fact, a proof along the lines of that of Theorem 2.8 [3] can be given to show that $W_2(\pi, \omega; Z[\tfrac{1}{2}]) = 0$ if $k_1 = k_2 = \dots = k_r = 2$. In this connection, we should note that here nontriviality in the eigenvalue decomposition of $W_2(\pi, \omega; Z[\tfrac{1}{2}])$ occurs only in those summands where all of the homomorphisms given by squaring the generator act as multiplication by -1 . But here, to complete the proof along the lines of 2.8 of [3], we need only observe that

$$\langle x_1, T_i T_j x_1 \rangle = -\langle T_i^3 x_1, T_j x_1 \rangle = -\langle -T_i x_1, T_j x_1 \rangle \\ = \langle T_i x_1, T_j x_1 \rangle = \dots = \langle T_i T_j x_1, x_1 \rangle = 0,$$

where it equals zero by skew-symmetry. The upshot of this is that $W_2(\pi, \omega; Z[\tfrac{1}{2}]) = \{0\}$ unless k_1 exceeds 2. By virtue of this, we assume that $k_1 > 3$, let T denote the generator of $Z_{2^{k_1}}$, and define the homomorphism (compare [2] and [3])

$$s: W_2^-(\pi, \omega; Z[\frac{1}{2}]) \rightarrow W_0^-(\pi, \omega; Z[\frac{1}{2}]), \quad (5)$$

given by the change of inner product

$$\langle x, y \rangle = (x, T^{2^{k_1-2}}y).$$

This procedure makes a skew-symmetric inner product symmetric and its inverse is provided by

$$(x, y) = \langle x, -T^{2^{k_1-2}}y \rangle.$$

Using the isomorphism (2), we are now able to compute $W_2(\pi, \omega; Z[\frac{1}{2}])$ by viewing the computation of $W_0(\pi, \omega; Z[\frac{1}{2}])$ in the light of these observations. Its dimension can be read immediately from (3) as

$$(k_r - m_1)2^{m_1 + \dots + m_s + r - 1} + (m_1 - m_2)2^{m_2 + \dots + m_s + r} \\ + \dots + (m_{s-1} - m_s)2^{m_s + r + s - 2} + (m_s - 2)2^{r + s - 1}.$$

Here, we let $m_0 = k_r$ and $m_{s+1} = 2$, enabling us to reformulate this expression as

$$2^{r-3} \sum_{i=0}^s (m_i - m_{i+1}) 2^i \prod_{j=1+i}^{s+1} 2^{m_j}.$$

This completes the proof of (b) of Theorem 2. The proof of (a) is easily inferred from the above discussion, taking heed of the fact that the change of inner product isomorphism (5) applies to the case with integer coefficients as well. We leave the details to the reader.

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