ON THE LOCALISATION SEQUENCE IN K -THEORY

VICTOR SNAITH¹

ABSTRACT. A description of the boundary map in Quillen's localisation sequence is given in terms of classifying spaces of categories. Using this description the low dimensional part of the localisation sequence for a Dedekind domain is shown to coincide with the algebraically defined Bass-Tate sequence.

0. Introduction. Let R be a ring with unit and $S \subset R$ a multiplicative set of central, nonzero divisors. Let R_S be the localised ring. If H is the exact category of finitely generated R -modules, M , having projective dimension < 1 and $M_S = 0$ there is a localisation sequence $[G-Q]$ ($q > 0$)

$$\dots \rightarrow K_{q+1}(R_S) \xrightarrow{\partial} K_q(H) \xrightarrow{a} K_q(R) \xrightarrow{b} K_q(R_S) \rightarrow \dots$$

The proof consists of utilising Quillen's Theorem B [Q, §1] to realise a and b by means of a categorical quasi-fibration. This gives nice categorical descriptions of a and b but not of ∂ . Using the techniques of [G-Q], [Q] it is shown (in §1.5) how to realise ∂ and a by means of a categorical quasi-fibration. In §2 we apply this to compute ∂ when R is a Dedekind domain with quotient field R_S .

1. Let R be a ring, $S \subset R$ is a multiplicative set of central nonzero divisors and H is the category of finitely generated R -modules, M , of projective dimension < 1 such that $M_S = 0$.

Let G denote the following category. Its objects are surjections $L \twoheadrightarrow M \oplus B$ with $L, B \in \mathbf{P}(R)$, the category of finitely generated projective R -modules, and $M \in H$. An arrow $(L' \twoheadrightarrow M' \oplus B') \rightarrow (L \twoheadrightarrow M \oplus B)$ is represented by an equivalence class of diagrams

$$\begin{array}{ccc} L' & \twoheadrightarrow & M' \oplus B' \\ \parallel & & \uparrow \\ L' & \twoheadrightarrow & M_1 \oplus B_1 \\ \downarrow & \square & \downarrow \\ L & \twoheadrightarrow & M \oplus B \end{array}$$

in which $\square$ denotes a pullback square and in which the vertical arrows are direct sums of arrows from H and $\mathbf{P}(R)$. Two such diagrams are equivalent if they are isomorphic by an isomorphism which is the identity on $M \oplus B$ and $M' \oplus B'$.

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Composition of two morphisms is accomplished by a pullback construction analogous to the composition in Quillen's Q -category [Q, §2].

The category, G , was introduced in [G-Q, p. 230] and there the following is shown.

1.1. LEMMA. *The functor $h: (L \twoheadrightarrow M \oplus B) \mapsto M$ induces a homotopy equivalence $h: G \xrightarrow{\sim} QH$.*

Now consider the functor $g: G \rightarrow QPR$ given by $g(L \twoheadrightarrow M \oplus B) = (B)$. This functor is fibred [G-Q]. That is, the functor $\Gamma: g^{-1}(B) \rightarrow B/g$, $\Gamma(L \twoheadrightarrow M \oplus B) = (L \twoheadrightarrow M \oplus B, 1_B)$, has an adjoint, γ . Therefore [Q, §1] $\Gamma: g^{-1}(B) \rightarrow B/g$ is a homotopy equivalence.

Let us fix $B \in \mathbf{P}(R)$ and consider $g^{-1}(B)$. It has objects $(L \twoheadrightarrow M \oplus B)$, and a morphism $(L' \twoheadrightarrow M' \oplus B) \rightarrow (L \twoheadrightarrow M \oplus B)$ is an equivalence class of diagrams as in the category G with the additional restriction that the endomorphisms of B in the diagram should be the identity.

Define a second category $\tilde{g}^{-1}(B)$ whose objects are pairs $(L \twoheadrightarrow M \oplus B, B \twoheadrightarrow L)$ in which the first entry is an object of G , and the composition $B \twoheadrightarrow L \twoheadrightarrow M \oplus B$ is the inclusion of the second summand. Morphisms are equivalence classes of commutative diagrams of the following form.

$$\begin{array}{ccccc}
 & & B & & \\
 & \swarrow & \uparrow & \searrow & \\
 L & \xrightarrow{\quad} & M \oplus B & & \\
 \parallel & & \downarrow 1_B & & \uparrow \alpha \oplus 1_B \\
 L & \xrightarrow{\quad} & M' \oplus B & & \\
 \downarrow & & \downarrow 1_B & & \downarrow \beta \oplus 1_B \\
 L_1 & \xrightarrow{\quad} & M_1 \oplus B & &
 \end{array}$$

Composition of morphisms is defined as in G and $g^{-1}(B)$.

1.2. PROPOSITION. *The functor $f: \tilde{g}^{-1}(B) \rightarrow g^{-1}(B)$ which forgets the morphism $B \twoheadrightarrow L$ is a homotopy equivalence.*

PROOF. Fix an object $(L_1 \twoheadrightarrow M_1 \oplus B)$ in $g^{-1}B$ and choose a morphism $B \twoheadrightarrow L_1$ so that $B \twoheadrightarrow L_1 \twoheadrightarrow M_1 \oplus B$ is the inclusion of B . We will now show that $f/(L_1 \twoheadrightarrow M_1 \oplus B)$ is contractible. For an object of this category is represented by a diagram of the following form.

$$\begin{array}{ccc}
 L & \xrightarrow{\quad} & M \oplus B \\
 \parallel & & \uparrow \alpha \oplus 1 \\
 L & \xrightarrow{\quad} & M' \oplus B \\
 \downarrow & & \downarrow \beta \oplus 1 \\
 L_1 & \xrightarrow{\quad} & M_1 \oplus B
 \end{array}$$

Hence the pullback square gives a natural morphism $B \twoheadrightarrow L$ from $B \twoheadrightarrow L_1$ and $B \subset M' \oplus B$. This gives a functor $a: (f/L_1 \rightarrow M_1 \oplus B) \rightarrow (1/L_1 \rightarrow M_1 \oplus B)$ where $1: g^{-1}(B) \rightarrow g^{-1}(B)$ is the identity functor. However forgetting $B \twoheadrightarrow L$ gives a left inverse to a . Thus $B(f/L_1 \rightarrow M_1 \oplus B)$ must be contractible since it is a factor in $B(1/L_1 \rightarrow M_1 \oplus B)$, and $1/L_1 \rightarrow M_1 \oplus B$ has a final object.

Let $I = \text{Iso } \mathbf{P}(R)$ the category of finitely generated projective R -modules and the isomorphisms. I acts on $g^{-1}(0)$ by

$$A + (L \twoheadrightarrow M) = (A \oplus L \xrightarrow{0+\pi} M).$$

The category $\langle I, g^{-1}(0) \rangle$ has the same objects as $g^{-1}(0)$, and a morphism $(L \rightarrow M) \rightarrow (L' \rightarrow M')$ is an isomorphism class of diagrams of the following form

$$\begin{array}{ccc} L' & \xrightarrow{\quad} & M' \\ \uparrow & \square & \uparrow \\ A \oplus L & \xrightarrow{\quad} & M'' \\ \parallel & & \downarrow \\ A \oplus L & \xrightarrow{\quad} & M \end{array}$$

Here $A \in \mathbf{P}(R)$ and an isomorphism of diagrams of the above type is required on $A \oplus L$ to be the direct sum of an isomorphism on A and one on L .

1.3. PROPOSITION. $\langle I, g^{-1}(0) \rangle$ is contractible.

PROOF. Direct sum makes $\langle I, g^{-1}(0) \rangle$ into an H -space, which is contractible by the proof of [G-Q, p. 227] as follows. The H -space is connected because $(L_i \twoheadrightarrow M_i)$ ($i = 1, 2$), maps to $(L_1 \oplus L_2 \twoheadrightarrow M_1 \oplus M_2)$ by the inclusion of the i th factor. Therefore we have only to show $\pi_i(B\langle I, g^{-1}(0) \rangle, 0) = 0$ ($i > 1$). However multiplication by two on these (abelian) homotopy groups is induced by the functor $F: (L \twoheadrightarrow M) \rightarrow (L \oplus L \twoheadrightarrow M \oplus M)$. The inclusion of the first summand gives a natural transformation from $\alpha: 1 \rightarrow F$ so that $BF \simeq 1$. Hence $2x = x$ for $x \in \pi_i(B\langle I, g^{-1}(0) \rangle, 0)$ and the group must be zero.

1.4. THEOREM. The square

$$\begin{array}{ccc} I^{-1}g^{-1}(0) & \xrightarrow{i} & I^{-1}G \\ \downarrow & & \downarrow g \\ * & \rightarrow & QP(R) \end{array}$$

is homotopy cartesian, where $i(A, L \twoheadrightarrow M) = (A, L \twoheadrightarrow M)$.

PROOF. By [Q, §1, Theorem 8] we must show that a morphism $0 \xleftarrow{\alpha} B' \xrightarrow{\beta} B$ induces base change homotopy equivalences

$$I^{-1}g^{-1}(0) \xrightarrow{I^{-1}\alpha^*} I^{-1}g^{-1}(B') \xrightarrow{I^{-1}\beta^*} I^{-1}g^{-1}(B)$$

when $I^{-1}\alpha^*, I^{-1}\beta^*$ are induced by α^*, β^* .

By §1.2 we may replace $g^{-1}(0)$ etc. by $\tilde{g}^{-1}(0)$. Also there are equivalences

$$\tilde{g}^{-1}(0) \xrightarrow{j_1} \tilde{g}^{-1}(B) \xrightarrow{j_2} \tilde{g}^{-1}(0)$$

given by $j_1(L \rightarrow M) = (L \oplus B \rightarrow M \oplus B)$ and

$$j_2 \left(\begin{array}{ccc} & B & \\ & \swarrow j & \\ L' & \xrightarrow{\pi} & M' \oplus B \end{array} \right) = (\text{coker } j \rightarrow M').$$

With this identification $\beta^*: \tilde{g}^{-1}(B) \rightarrow \tilde{g}^{-1}(B')$ becomes the identity on $\tilde{g}^{-1}(0)$ because β^* is pullback by β .

$$\begin{array}{ccccc} B' & \xrightarrow{j'} & L' & \longrightarrow & M \oplus B' \\ \beta \downarrow & & \downarrow & \square & \downarrow 1 \oplus \beta \\ B & \xrightarrow{j} & L & \longrightarrow & M \oplus B \end{array}$$

Furthermore α^* is given by sending $(L \rightarrow M \oplus B')$ to $(L \rightarrow M \oplus B' \xrightarrow{\pi_2} M)$ which becomes the action of B' on $\tilde{g}^{-1}(0)$. However by Proposition 1.3 and [G-Q, p. 223] multiplication by B' on $I^{-1}\tilde{g}^{-1}(0) \simeq I^{-1}g^{-1}(0)$ is a homotopy equivalence.

1.5. COROLLARY. *There is a commutative diagram ($q > 0$)*

$$\begin{array}{ccccccc} \pi_{q+1}(BI^{-1}g^{-1}(0)) & \xrightarrow{i} & \pi_{q+1}(BQH) & & & & \\ \cong \downarrow & & \parallel & & & & \\ \cdots \rightarrow K_{q+1}R_S & \longrightarrow & K_q(H) & \longrightarrow & K_q(R) & \longrightarrow & K_q(R_S) \longrightarrow \cdots \end{array}$$

in which the row is the localisation sequence [G-Q, p. 233] and i is induced by the functor $i(A, L \rightarrow M) = (M)$.

PROOF. From the derivation of the localisation sequence [G-Q, pp. 229–233] we see that the equivalences $I^{-1}G \simeq I^{-1}QH \simeq QH$ transform the functor g of §1.4 into the map inducing $K_*(H) \rightarrow K_*(R)$. Hence Theorem 1.4 extends to the left the (quasi-) fibration sequence $BQH \rightarrow BQP(R) \rightarrow BQP(R_S)$ and the result follows from the uniqueness, up to homotopy, of this extension.

1.6. REMARK. Observe that Theorem 1.4 amounts to the assertion that $BI^{-1}g^{-1}(0)$ equals $(K_0R/(\text{im}(K_0H))) \times BGLR_S^+$.

2. Throughout this section set $R = A$, a Dedekind domain, with quotient field $F = R_S$. In this case devissage applied to K_*H transforms the localisation sequence into the form

$$\cdots \rightarrow K_iA \xrightarrow{b} K_iF \xrightarrow{\partial} \bigoplus_{P \triangleleft_{\max} A} K_{i-1}(A/P) \xrightarrow{a} K_{i-1}A \rightarrow \cdots$$

Therefore we have $K_1 F = F^*$ and $K_0(A/P) = Z(P)$, a copy of the integers. These identifications yield a homomorphism

$$\partial: F^* \rightarrow \bigoplus_{P \triangleleft_{\max} A} Z(P). \quad (2.1)$$

We will apply §§1.4–1.5 to prove the following result (asserted by Quillen [Q, §7, Remark 5.17]). This result did not appear in [G-Q] as originally promised in [Q, *ibid.*].

2.2. THEOREM. *The homomorphism of (2.1) is given by $\partial(x) = \sum \nu_P(x)$ where $x \in F^*$ and ν_P is the P -adic valuation.*

PROOF. Firstly set $J = \text{Iso } \mathbf{P}(F)$. Now let us give a functor $f: I^{-1}g^{-1}(0) \rightarrow J^{-1}J$ which is a homotopy equivalence on base-point components (both equal to $BGLF^+$). Having done this we will be able to use the description given in [G-Q, p. 224] of the loop in $\pi_1(BGLF^+) = F^*$ corresponding to $x \in A \subset F$ and thereby evaluate $\partial(x)$.

Assigning $(\text{Ker } \pi \triangleleft L)$ to $(\pi: L \twoheadrightarrow M)$ gives an equivalence of categories with I -action between $g^{-1}(0)$ and the category, $\mathcal{L}$, of admissible layers in $\mathbf{P}(A)$. The functor $h(L_0 \triangleleft L_1) = L_1 \otimes_A F$ is a homotopy equivalence of categories with I -action, $h: \mathcal{L} \rightarrow J$. This is because $(L \otimes_A F)/h$ equals the category of A -lattices which is called $X(L \otimes_A F)$ in [Q, §7, Remark 5.17] and is shown there to be contractible. Hence $I^{-1}g^{-1}(0) \rightarrow I^{-1}J$ is a homotopy equivalence.

Now consider $I^{-1}J \rightarrow J^{-1}J$ induced by $I \rightarrow J$. This induces a homology isomorphism on base-point components of the classifying spaces, by [G-Q, p. 222]. However direct sum makes $BI^{-1}J$ and $BJ^{-1}J$ into H -spaces (similarly base-point components) and so the map on base-point components is a homotopy equivalence.

Set f equal to the composite $I^{-1}g^{-1}(0) \rightarrow I^{-1}J \rightarrow J^{-1}J$.

Now let $x \in A$. In $g^{-1}(0)$ we have two morphisms

$$\begin{array}{ccc} j: & A & \twoheadrightarrow 0 \\ & \parallel & \uparrow \\ & A & \twoheadrightarrow A/xA \\ & \parallel & \parallel \\ & A & \twoheadrightarrow A/xA \end{array} \quad \text{and} \quad \begin{array}{ccc} i: & A & \twoheadrightarrow 0 \\ & \parallel & \parallel \\ & A & \twoheadrightarrow 0 \\ & \downarrow x & \downarrow \\ & A & \twoheadrightarrow A/xA \end{array}$$

resulting in a loop $(1_A, i^{-1} \circ j) \in \pi_1(BI^{-1}g^{-1}(0); (A, A \twoheadrightarrow 0))$. The functor f takes this loop to the loop which is shown in [Q, §2, Theorem 1] to determine $[A/xA] \in K_0 H = \bigoplus_{P \triangleleft_{\max} A} K_0(A/P)$. By definition of the last (devissage) isomorphism [Q, §5, Corollary 1] A/xA corresponds to $\sum \nu_P(x)$ as required.

2.3. REMARK. The formula of §2.2 agrees with that of [M, p. 123] for the algebraically defined coboundary in the localisation sequence of Bass and Tate.

Recall that the algebraically defined coboundary [M, p. 133]

$$\partial: K_2 F \rightarrow \bigoplus_{P \triangleleft_{\max} A} (A/P)^* \cong \bigoplus_{P \triangleleft_{\max} A} K_1(A/P) \quad (2.4)$$

is given by the tame symbol. If $x, y \in F^*$ have Steinberg symbol $\{x, y\} \in K_2F$ the P -coordinate of $\partial\{x, y\}$ is given by [M, §11.5]

$$\partial\{x, y\}_P = (-1)^{v_P(x)v_P(y)} x^{v_P(y)} y^{-v_P(x)}. \quad (2.5)$$

2.6. THEOREM. *The homomorphism of (2.4) is equal to the tame symbol whose P -coordinate is given by (2.5).*

PROOF. Recall that K_2F is generated by Steinberg symbols $\{x, y\}$ ($x, y \in F^*$) and that $\{, \}: K_1F \otimes K_1F \rightarrow K_2F$ is equal to the topologically defined K -theory product given by Loday [L, §2.2.3]. Furthermore the localisation sequence derived in [G-Q, p. 233] is a sequence of left K_*A -modules under Loday's product.

Also by naturality of the localisation sequence we can reduce to the local case. In this case let v be the valuation associated to the maximal ideal. Let $x, y \in F^*$ satisfy $v(x) = n$, $v(y) = m$. Choose $z \in F^*$ so that $v(z) = 1$. We may write $x = z^n a$, $y = z^m b$ with $a, b \in A^*$.

We will write K_2F additively and $K_1(A/P)$ multiplicatively.

Since $0 = \{z, 1 - z\}$ the bilinearity and skew-symmetry of the symbol yields $0 = \{z, z^{-1}\} + \{z, 1 - z\} + \{z, -1\}$. Therefore $\{z, z\} = \{z, -1\}$ and $\partial\{z, z\} = \partial\{-1, z\}^{-1} = ((-1)^{v(z)})^{-1} = -1$, by the K_1A -module structure and Theorem 2.2.

Similarly we compute in general that

$$\begin{aligned} \partial\{x, y\} &= \partial(mn\{z, z\} + \{a, y\} - \{b, x\}) \\ &= (-1)^{mn} a^{v(y)} b^{-v(x)} \end{aligned}$$

which is the expression given in (2.5).

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF WESTERN ONTARIO, LONDON, ONTARIO N6A 5B9, CANADA